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Duality and the Geometry of the Income and Substitution Effects

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Abstract

The geometry of the Hicks-Slutsky income and substitution effect is framed in terms of the consumer's expenditure function and expenditure equation and thus can be studied without resorting to the indifference map of a direct utility function.

Key words: duality; income/substitution effect; expenditure function

JEL classification: D1; D6

1. Introduction

An important analytical methodology for solving the consumer equilibrium problem is the *dual approach* [see for instance Deaton and Muellbauer (1984), McKenzie (1957), and Diewert (1974, 1981). For a similar though production-oriented discussion of duality see Fuss and McFadden (1978) and Shephard (1970)]. That is, instead of seeking the income constrained maximum of a direct utility function, emphasis has alternatively been placed upon dealing with its *mirror image* problem, namely the derivation of the expenditure function as the utility constrained minimum of total expenditure. Indeed, for empirical purposes, the expenditure approach to demand estimation has experienced great popularity and exhibited marked success [Deaton and Muellbauer (1984)].

The expenditure function serves as an alternative to the direct utility function as a representation of preferences and is of considerable importance since it greatly simplifies the measurement and computation of welfare changes, expedites the study of economic index numbers, and enables us to derive the basic propositions of consumer demand theory. However, in spite of all the important advances in economic theory that have been facilitated by the use of the expenditure function, we seem to be inexorably wedded to the traditional use of indifference curves when it comes to describing the income and substitution effects of a price change. A problem with the traditional approach is that it relies upon special assumptions pertaining to preferences and to tangency conditions (e.g., strict convexity of preferences and sufficient

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smoothness of the direct utility function so that differentiation can be performed).

In what follows the geometry of the Hicks-Slutsky decomposition of the gross effect of an own-price increase into a substitution effect and an income effect is framed in terms of the consumer's expenditure function and expenditure equation. The advantage of this dual approach is that the income and substitution effect can be studied without resorting to the indifference map of a direct utility function. Indeed, the "law of demand" directly follows from the shape of the expenditure function and not from the properties of indifference curves. Moreover, working with the expenditure function enables us to readily evaluate the welfare implications of a price change; we can easily determine, for instance, how much extra purchasing power a household would need to compensate for a rise in the price of given commodity.

2. Demand, Expenditure, and Indirect Utility Functions

Let $\boldsymbol{Q}$ be an n by 1 commodity vector with $\boldsymbol{p}$ the corresponding n by 1 vector of commodity prices, where Q_i is the ith component of $\boldsymbol{Q}$ and p_i is the ith component of $\boldsymbol{p}$, $i = 1, 2, \ldots, n$. Given constant commodity prices, a constant money income level M, and a direct well-behaved utility function $u = v(\boldsymbol{Q})$, the consumer's choice problem is to typically determine quantities Q_i which

$$\text{maximize } u = v(\boldsymbol{Q}) \text{ subject to } \Sigma\, p_i\, Q_i = M. \tag{1}$$

The solution to this problem is the system of Marshallian or constant money income demand functions $g^i(\boldsymbol{p}, M)$, $i = 1, 2, \ldots, n$. However, for fixed commodity prices and a given level of utility u, the problem dual to (1) has the consumer choose Q_i which

$$\text{minimize } M = \Sigma\, p_i\, Q_i \text{ subject to } v(\boldsymbol{Q}) = u. \tag{2}$$

The solution to this dual problem is the set of Hicksian or compensated demand functions $h^i(\boldsymbol{p},u)$, $i = 1, 2, \ldots, n$. The functions g^i are assumed to be homogenous of degree zero in prices and income while the h^i are taken to be homogenous of degree zero in prices alone.

One important feature of these demand functions is that they add up. In particular, for the compensated demand functions, the total value of the equilibrium Hicksian demands exhausts total expenditure or $\sum_{i=1}^{n} p_i h^i = M$. Since h^i has as its arguments prices $\boldsymbol{p}$ and total utility u (= constant), it follows that total expenditure is also a function of these variables and thus can be represented by the *expenditure function* $M = c(\boldsymbol{p}, u)$. Here c indicates the minimum monetary expenditure needed to attain utility level u at the given prices $\boldsymbol{p}$. A key property of the expenditure function is that it possesses the derivative property (*Shephard's lemma*) [Shephard (1970)]: where they exist, the partial derivatives of c with respect to p_i (≥ 0) yield the equilibrium compensated demand functions $\partial c(\boldsymbol{p},u)/\partial p_i = h^i(\boldsymbol{p},u)$. [Other properties of the expenditure function may be found in Deaton and Muellbauer (1984).]

In a similar vein, substituting the ordinary or Marshallian demands g^i into the direct utility function renders the *indirect utility function* $u = v(g^1, \ldots, g^n) = k(\boldsymbol{p},u)$.

This expression indicates the maximum level of utility attainable for a total expenditure of M at given prices. While the salient features of the indirect utility function can be found in Deaton and Muellbauer (1984), it is important to note that if k is differentiable, then the ordinary demand functions are obtainable via *Roy's identity*, i.e., $g^i(\boldsymbol{p},u) = -(\partial k/\partial p_i)/(\partial k/\partial M)$. The relationship between the expenditure function and the indirect utility function is referred to as *utility-expenditure duality*, so called because, at the set of prices for which c yields the minimum expenditure required to achieve u and k yields the maximum utility attainable given M, we have $c(\boldsymbol{p}, k(\boldsymbol{p}, M)) \equiv M$ or $k(\boldsymbol{p}, c(\boldsymbol{p}, u)) \equiv u$.

3. Hicks-Slutsky Decomposition of a Price Change

For purposes of exposition, we shall henceforth assume that the consumer purchases only two commodities in the quantities Q_1 and Q_2 with their prices denoted as p_1 and p_2 respectively. In this regard two identities hold:

$$c(p_1, p_2, u) \equiv c(p_1, p_2, k(p_1, p_2, M)) \equiv M,$$

(the minimum expenditure necessary to reach utility $k(p_1,p_2,M)$ is M); and for p_2 constant,

$$c(p_1+ \Delta p_1, p_2, u + \Delta u) \equiv c(p_1+ \Delta p_1, p_2, k(p_1+ \Delta p_1, p_2, M)),$$

where M is the original level of expenditure at (p_1, p_2, u).

Given $p_2 = p_2^0$ = constant, the linearization of the surface $M = c(p_1, p_2^0, u)$ at an arbitrary point (p_1, p_2^0, u) is, by virtue of the preceding identities,

$$c(p_1+ \Delta p_1, p_2^0, k(p_1 + \Delta p_1, p_2^0, M)) = c(p_1, p_2^0, k(p_1, p_2^0, M))+ \frac{\partial c}{\partial p_1}\Delta p_1 + \frac{\partial c}{\partial k}\frac{\partial k}{\partial p_1}\Delta p_1,$$

or, with the level of total expenditure constant and $\Delta p_1 \neq 0$,

$$0 = \frac{\partial c}{\partial p_1}\Delta p_1 + \frac{\partial c}{\partial k}(-g^1\frac{\partial k}{\partial M})\Delta p_1 = \frac{\partial c}{\partial p_1} - g^1. \tag{3}$$

(via Roy's identity). A differentiation of (3) with respect to p_1 renders

$$\frac{\partial g^1}{\partial p_1} = \frac{\partial^2 c}{\partial p_1^2} + \frac{\partial^2 c}{\partial p_1 \partial k}\frac{\partial k}{\partial p_1}, \tag{3.1}$$

or, by virtue of the derivative property,

$$\frac{\partial g^1}{\partial p_1} = \frac{\partial h^1}{\partial p_1} + \frac{\partial h^1}{\partial u}\frac{\partial u}{\partial p_1}, \tag{3.2}$$

where $\partial u / \partial p_1$ is the partial derivative of the indirect utility function with respect to p_1.

Here the first term on the right-hand-side of (3.2) is the compensated price derivative (u is held constant) or the own-price substitution effect of a compensated price change; the second term represents the income effect of a price change (compensation is adjusted so as to keep relative prices constant).

In terms of discrete changes (Δp_1 small), (3.2) can be expressed as

$$\begin{aligned}\Delta Q_1 &\approx \frac{\partial h^1}{\partial p_1}\Delta p_1 + \frac{\partial h^1}{\partial k}\frac{\partial k}{\partial p_1}\Delta p_1 \\ &= \frac{\partial h^1}{\partial p_1}\Delta p_1 - Q_1\frac{\partial h^1}{\partial k}\frac{\partial k}{\partial M}\Delta p_1 ,\end{aligned} \tag{3.3}$$

using Roy's identity.

How does (3.2) compare with the traditional specification of the income and substitution effect? Since, at equilibrium, $h^1 = g^1$ and $u = k$, it follows from another application of Roy's identity that (3.2) can be rewritten as

$$\begin{aligned}\frac{\partial g^1}{\partial p_1} &= \frac{\partial h^1}{\partial p_1} + \frac{\partial g^1}{\partial k}(-Q_1\frac{\partial k}{\partial M}) \\ &= \frac{\partial h^1}{\partial p_1} - Q_1\frac{\partial g^1}{\partial M} .\end{aligned} \tag{4}$$

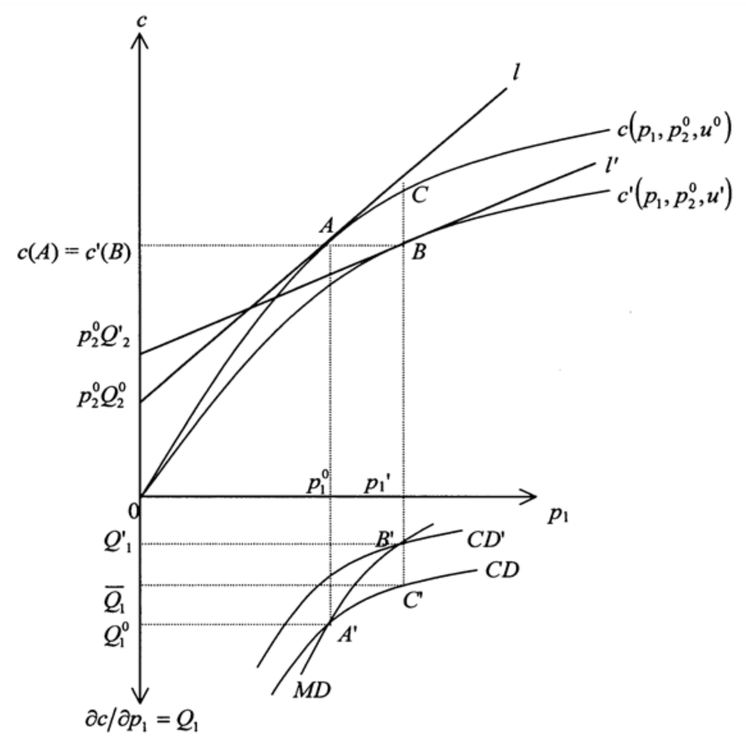
(the so-called *Hicks - Slutsky equation*). The usual interpretation of (4) applies: the total effect of an own-price change on ordinary demand is decomposable into a substitution effect and an income effect. Furthermore, with Δp_1 small, (4) can be modified as

$$\Delta Q_1 \approx \frac{\partial h^1}{\partial p_1}\Delta p_1 - Q_1\frac{\partial g^1}{\partial M}\Delta p_1. \tag{4.1}$$

4. The Geometry of the Income and Substitution Effects

To cast the geometry of this decomposition in terms of the properties of the expenditure function, let us consider panel (a) of Figure 1 wherein we initially consider a given expenditure function c along with the fixed budget line $M = p_1 Q_1^0 + p_2^0 Q_2^0$ (depicted as l) with argument p_1 and slope Q_1^0 tangent to c at point A, where u^0 and $p_2^0 Q_2^0$ are constant. In panel (b) we have, by Shephard's lemma, the compensated demand function $(CD)\,\partial c/\partial p_1 = h^1$. At a price of p_1^0 we have, at point A, $c(p_1^0, p_2^0, k(p_1^0, p_2^0, M)) \equiv M$ while at A' the quantity Q_1^0 satisfies both the Hicksian as well as the ordinary or Marshallian demand relationships, i.e., $g^1(p_1^0, p_2^0, c(p_1^0, p_2^0, u^0)) = h^1(p_1^0, p_2^0, u^0)$ so that u in the compensated demand curve equals the maximum utility level represented by k and the minimum expenditure level depicted by c equals the fixed expenditure level M in the ordinary demand function.

Fig. 1.



(a) pure substitution effect ($A \to C$);
pure income effect ($C \to B$);
total effect ($A \to B$).

(b) pure substitution effect $Q_1^0 - \overline{Q}_1(A' \to C')$;
pure income effect ($C' \to B'$);
total effect $Q_1^0 - Q_1'(A' \to B')$;

Let the price of Q_1 increase to p_1' (Δp_1 small). This is reflected by the new budget line $M = M' = p_1 Q_1' + p_2^0 Q_2'$ (denoted l') with slope Q_1', where Q_2' is the new level of Q_2 purchased. Since at p_1' we have $u' = k(p_1', p_2^0, M) < u^0 = k(p_1^0, p_2^0, M)$, it follows that with the decrease in the optimal utility level, M' is tangent to a lower expenditure function $c'(p_1', p_2^0, u')$ at B (horizontally aligned with A since total expenditure is constant). At this point $c'(p_1', p_2^0, k(p_1', p_2^0, M)) \equiv M$. Moreover, in panel (b), the new compensated demand function corresponding to expenditure function c' is CD' and, at point B', $g^1(p_1', p_2^0, c(p', p_2^0, u')) \equiv h^1(p', p_2^0, u')$. In this regard A' and B' depict two points on the same Marshallian demand function (*MD*). So with the increase in the price of Q_1 from p_1 to p_1', we observe a movement down *MD* from A' to B' as Q_1 drops to its new level Q_1'. To eliminate the decrease in real income resulting from a price increase, let us compensate the household by an amount *CB* sufficient to restore its original utility level u^0 at the new set of market prices. But this implies that the new quantity demanded $\overline{Q}_1$ lies on the original compensated demand function *CD* since the household is able to enjoy the *original* real income level at the *new* market price set.

Thus the movement from *A* to *B* (or from *A′* to *B′*) occurs in two steps. First, there exists a *pure substitution effect* of a price increase which reflects a change in the household's purchase mix due to a change in relative prices and which involves a movement from *A* to *C* along the expenditure function (or from *A′* to *C′* along the compensated demand function). Second, the movement from *C* to *B* (or from *C′* to *B′*) constitutes the *pure income effect* of a price increase (compensation is withdrawn) and is indicative of the effect on the commodity mix of a change in the household's absolute purchasing power. In terms of equation (3.3) (or (4.1)), $\Delta Q_1 \approx Q_1^0 - Q_1' = (Q_1^0 - \overline{Q}_1) + (\overline{Q}_1 - Q_1')$. Note that in panel (b) of Figure 1 the compensated demand curve is downward sloping (*c* is concave in p_1) and thus the own-price substitution effect can never be positive. The income effect, as usual, will be either negative or positive, depending upon whether Q_1 is a normal or inferior good respectively. In fact, if Q_1 is normal $(\partial Q_1/\partial M > 0)$, then in the typical price-quantity diagram, $|\partial h^1/\partial p_1| > |\partial g^1/\partial p_1|$, i.e., in absolute terms, the compensated demand function is steeper than the ordinary demand function at any Q_1 level. Clearly panel (b) in Figure 1 is consistent with this result.

5. A Parametric Example

Given the strictly quasi-concave utility function $u=Q_1Q_2$ and the linear budget constraint $M=p_1Q_1 + p_2Q_2$, the consumers primal problem appears as

maximize $u = Q_1Q_2$ subject to $M-p_1Q_1-p_2Q_2 = 0$.

From the Lagrangian function $L(Q_1,Q_2, \lambda) = Q_1Q_2 + \lambda(M - p_1Q_1 - p_2Q_2)$ (where λ is the undetermined Lagrange multiplier), the first-order conditions $L_1 = L_2= L_\lambda= 0$ yield the optimal solution or set of Marshallian demands and λ:

$$\begin{aligned} Q_1 &= g^1(p_1, p_2, M) = M/2p_1, \\ Q_2 &= g^2(p_1, p_2, M) = M/2p_2, \\ \lambda &= M/2p_1p_2 . \end{aligned} \tag{5}$$

(Here the second-order condition for a constrained maximum is satisfied at the point where the first-order conditions hold since the utility function is strictly quasi-concave.)

Dually, let us minimize the cost of obtaining *a* fixed utility level *u* or

minimize $p_1Q_1+ p_2 Q_2$ subject to $u - Q_1Q_2 = 0$.

The Lagrangian for this problem is $M(Q_1,Q_2,\mu) = p_1Q_1 + p_2Q_2 + \mu(u - Q_1Q_2)$, where μ is the undetermined Lagrange multiplier. Now, from the first-order conditions $M_1 = M_2 = M_\mu = 0$, we may solve for the Hicksian demands and μ:

$$Q_1 = h^1 (p_1, p_2, u) = (up_2/p_1)^{1/2},$$
$$Q_2 = h^2 (p_1, p_2, u) = (up_1/p_2)^{1/2}, \tag{6}$$
$$\mu = \lambda^{-1}.$$

Upon substituting the Marshallian demands into the direct utility function $u = Q_1Q_2$ enables us to find the indirect utility function

$$k(p_1, p_2, M) = \frac{1}{4} p_1^{-1} p_2^{-1} M^2. \tag{7}$$

Moreover substituting the Hicksian demands into the budget equation $M = p_1Q_1 + p_2Q_2$ results in the expenditure function

$$c(p_1, p_2, u) = 2(p_1 p_2 u)^{1/2}. \tag{8}$$

From (3.1) it is easily verified that

$$\frac{\partial g^1}{\partial p_1} = \frac{\partial^2 c}{\partial p_1^2} + \frac{\partial^2 c}{\partial k \partial p_1} \frac{\partial k}{\partial p_1}.$$

$$= -\frac{1}{4} p_1^{-2} M - \frac{1}{4} p_1^{-2} M = -\frac{1}{2} p_1^{-2} M; \tag{9}$$

[substitution effect] [income effect] [total effect]

and from (4), it is also true that

$$\frac{\partial g^1}{\partial p_1} = \frac{\partial h^1}{\partial p_1} - Q_1 \frac{\partial g^1}{\partial M} = -\frac{1}{2} p_1^{-2} M, \tag{10}$$

where Q_1 is determined from the Marshallian demand function g^1. In terms of discrete changes (Δp_1 small), the right-hand sides of (3.3) and (4.1) are each expressible as

$$\Delta Q_1 \approx -\frac{1}{2} p_1^{-2} M \, \Delta p_1,$$

as anticipated from equations (9) and (10).

It is instructive to isolate the amount of *Hicksian compensation* involved in the movement from point A to point C in panel (a) of Figure 1 or in the movement from point A' to point C' in panel (b) of the same. Specifically, we are interested in specifying the amount of money that makes the original level of utility obtainable under a change in the price of commodity 1. More formally, Hicksian compensation may be determined in the following fashion: the value of compensated demand for commodity 1 when its price changes by Δp_1 is

$$g^1(p_1 + \Delta p_1, p_2, M+\Delta M) \equiv g^1(p_1 + \Delta p_1, p_2, c(p_1 + \Delta p_1, p_2, u)), \tag{11}$$

where u is the original level of utility prevailing at (p_1, p_2, M). Expanding (11) linearly near (p_1, p_2, u), with Δp_1 small, yields

$$g^1(p_1 + \Delta p_1, p_2, c(p_1 + \Delta p_1, p_2, u)) = g^1(p_1, p_2, c(p_1, p_2, u)) + \frac{\partial g^1}{\partial p_1}\Delta p_1 + \frac{\partial g^1}{\partial c}\frac{\partial c}{\partial p_1}\Delta p_1. \tag{11.1}$$

From (9) we have

$$\frac{\partial g^1}{\partial p_1} = -\frac{1}{2}p_1^{-2}M\ ,$$

while from (8) we obtain

$$\frac{\partial g^1}{\partial c}\frac{\partial c}{\partial p_1} = \frac{1}{4}p_1^{-2}M\ ,$$

(given that $c \equiv M$ at equilibrium). Hence (11.1) becomes

$$g^1(p_1 + \Delta p_1, p_2, c(p_1 + \Delta p_1, p_2, u)) - g^1(p_1, p_2, c(p_1, p_2, u)) = -\frac{1}{4}p_1^{-2}M\ \Delta p_1, \tag{11.2}$$

the own-price substitution effect of a compensated price change depicted by the movement along curve CD from A' to C'.

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