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Measuring Scale Efficiency Change Using a Translog Distance Function

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Abstract

We derive a measure of scale efficiency change in the case of a translog output-oriented, multi-input, multi-output distance function without imposing restrictions in previous analogous works by Ray (1998, 1999) and Balk (2001). This may be particularly useful in productivity analysis based on the Malmquist productivity index.

Key words: most productive scale size (MPSS); scale efficiency change; translog function

JEL classification: B41; D20; D21

1. Introduction

In productivity analysis and especially according to the popular Malmquist index concept, productivity growth is defined as the product of technical efficiency change, technical change, and scale efficiency change. Ray (1998) developed measures than can be directly estimated from a translog production frontier of input- and output-oriented scale efficiency in the case of a single output. Later (1999), he extended this approach to measure scale efficiency in a multi-input, multi-output distance function. With the scale efficiency measurement in mind and using a translog distance function, he focused on a specific point of the constant returns to scale (CRS) frontier, namely the most productive scale size (MPSS), and thus his approach is irrespective of linearity. However, Ray has not introduced time, and thus scale, efficiency change in his analysis, and consequently productivity growth is not estimable, at least directly, through his approach.

Balk (2001) introduced a measure of scale efficiency change in the case of a translog distance function, which is the most common functional form in empirical

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work. His approach is developed in the context of CRS technology, named cone technology, which arises from the linear expansion of different input-output mixes.

In this paper, following both Ray (1998, 1999) and Balk (2001), we develop a measure of scale efficiency change derived from a full translog output-oriented, multi-input, multi-output distance function using MPSS as the reference point.

2. Theoretical Background

Generally speaking, a firm is conceived as an entity that transforms inputs into outputs. The input quantities can be represented as a K -dimensional vector of non-negative real values $\mathrm{x}_k^t = \{x_1^t, x_2^t, \ldots, x_k^t\}$, $\mathrm{x}_k^t \in \Re_+^K$, which are transformed into output quantities represented by the M -dimensional vector of non-negative real values $\mathrm{y}_m^t = \{y_1^t, y_2^t, \ldots, y_M^t\}$, $\mathrm{y}_m^t \in \Re_+^M$. The superscript t indicates that the corresponding variable or parameter is reviewed in period t . Accordingly, therefore, we follow the usual definitions of the production possibility set $T^t(y)$, technology set $L^t(y)$, and the output set $P^t(x)$:

$$T^t(y) \equiv \left\{ \left(x_k^t, y_m^t \right) : x_k^t \in \Re_+^K,\ y_m^t \in \Re_+^M,\ x_k^t \text{ can produce } y_m^t \right\},$$
$$L^t(y) \equiv \left\{ x_k^t : \left(x_k^t, y_m^t \right) \in T^t \right\},$$
$$P^t(x) \equiv \left\{ y_k^t : \left(x_k^t, y_m^t \right) \in T^t \right\}.$$

In addition, the following usual relations hold (Färe and Primont, 1995):

$$T^t(y) : P^t \times P^t \to \Re_+^N,\ L^t(y) \subset T^t \text{, and } P^t(x) \subset T^t .$$

The output distance function is defined as $D_O^t(x, y) \equiv \inf\{\delta > 0 : y^t / \delta \in P^t(x)\}$ and forms, along with the technology set, the distance space $\left\langle P^t, D_O(x^t, y^t) \right\rangle$. For simplicity the output distance function is denoted $D(x^t, y^t)$. An input-output bundle (x^t, y^t) is a feasible production plan if the output vector is producible from the input vector.

Following Banker (1984), we can define the input-output pair $(x_*^t, y_*^t) = (k_1^t x^t, k_2^t y^t)$ as the MPSS if:

$$\frac{k_2^t}{k_1^t} \geq \frac{k_2''^t}{k_1''^t},\ \forall (k_1''^t, k_2''^t) \in \Re^2 \text{ and } (k_1''^t x, k_2''^t y) \in T^t . \tag{1}$$

In the case that the technology exhibits CRS, and taking into account the properties of distance functions (Färe and Primont, 1995; Färe et al., 1994), the following relationship holds:

$$\sum_i \frac{\partial \ln D^{i,t}}{\partial \ln x^{i,t}} + \sum_i \frac{\partial \ln D^{i,t}}{\partial \ln y^{i,t}} = 0 .$$

Informally speaking, at the MPSS, the sum of the output and input elasticities of the input distance function must equal zero.

3. Scale Efficiency Change for an Output-Oriented Translog Distance Function

Scale efficiency (SE) measures a firm's productivity at a given point with respect to what it could accomplish if it operated at the MPSS, where the average productivity reaches a maximum level. In that sense, SE can be seen as the ratio of average productivity at any observed level of input use to the maximum average productivity attained at the MPSS point. But in the multi-input, multi-output case the concept of average productivity is inappropriate, whereas the notion of radial changes in the input bundle allows us to define radial average productivity (RAP) at the input-output bundle (x^t, y^t) in comparison to the MPSS input-output bundle (x_*^t, y_*^t) . Based on the notion of RAP , SE is defined as:

$$SE\left(x^t\right)=\frac{RAP\left(x^t\right)}{RAP\left(x_*^t\right)}, \tag{2}$$

where (x^t, y^t) is any actual and observed input-output bundle and (x_*^t, y_*^t) is the corresponding bundle at the MPSS. Subsequently, in order to measure scale efficiency change (SEC) between periods t and $t+1$, we use the definition:

$$SEC\left(x^{t+1}, x^t, y\right)=\frac{SE\left(x^{t+1}, \bar{y}\right)}{SE\left(x^t, \bar{y}\right)}=\frac{\breve{D}\left(x^{t+1}, \bar{y}\right)\Big/D\left(x^{t+1}, \bar{y}\right)}{\breve{D}\left(x^t, \bar{y}\right)\Big/D\left(x^t, \bar{y}\right)}, \tag{3}$$

where $\breve{D}_0^t\left(x^t, y^t\right)$ denotes the value of distance function under a virtual technology that coincides with CRS, which is the so-called cone technology (Balk, 2001). The input bundles in periods t and $t+1$ are denoted x^t and x^{t+1} , respectively.

Assume that for the i th firm, $i=1,2,\ldots,I$, in the t th period, $t=1,2,\ldots,T$, the output-oriented distance function is approximated by the translog function:

$$\begin{aligned}\ln D^{i,t}\left(x^{i,t}, y^{i,t}, R\right) &= a_0+\sum_{k=1}^{K} a_k \ln x_k^{i,t}+\frac{1}{2}\sum_{k=1}^{K}\sum_{l=1}^{K} a_{kl} \ln x_k^{i,t} \ln x_l^{i,t} \\ &+\sum_{k=1}^{K}\sum_{m=1}^{M}\delta_{km} \ln x_k^{i,t} \ln y_m^{i,t}+\sum_{m=1}^{M}\beta_m \ln y_m^{i,t}+\frac{1}{2}\sum_{m=1}^{M}\sum_{n=1}^{M}\beta_{mn} \ln y_m^{i,t} \ln y_n^{i,t} \\ &+\gamma_1 R+\frac{1}{2}\gamma_2 R^2+\sum_{k=1}^{K} n_k R \ln x_k^{i,t}+\sum_{m=1}^{M}\mu_m R \ln y_m^{i,t},\end{aligned} \tag{4}$$

where R is a time trend, which captures technical change. The restrictions $\sum_{m=1}^{M}\beta_m=1$, $\sum_{n=1}^{M}\beta_{mn}=0$, $\sum_{m=1}^{M}\mu_m=0$, $\sum_{m=1}^{M}\delta_{km}=0$ must hold. On the other hand, the translog output distance function that arises under cone technology $\breve{D}^{i,t}\left(x_*^{i,t}, y_*^{i,t}, R\right)$ results simply by replacing $x^{i,t}$ and $y^{i,t}$ in (4) with x_*^t and y_*^t , respectively. (Hereafter the superscript i is suppressed.) Vectors with superscript

(x_*^t, y_*^t) represent output-input combinations for the i th firm. Taking into consideration that $(x_*^t, y_*^t) = (k_1^t x^t, k_2^t y^t)$ and using the explicit form of the translog distance function given in (4), the distance function $\breve{D}^t\left(x_*^t, y_*^t, R\right)$ becomes:

$$
\begin{aligned}
\ln \breve{D}^t\left(x_*^t, y_*^t, R\right) &= \ln D^t\left(x^t, y^t, R\right) + \sum_{k=1}^{K} a_k \ln k_1^t \\
&+ \sum_{k=1}^{K} a_k \ln k_1^t + \frac{1}{2}\sum_{k=1}^{K}\sum_{l=1}^{K} a_{kl}\left[\left(\ln x_k^t \ln k_1^t\right) + \left(\ln x_l^t \ln k_1^t\right) + \left(\ln k_1^t\right)^2\right] \\
&+ \sum_{k=1}^{K}\sum_{m=1}^{M} \delta_{km}\left[\left(\ln x_k^t \ln k_1^t\right) + \left(\ln y_m^t \ln k_2^t\right) + \left(\ln k_1^t \ln k_2^t\right)\right] + \sum_{m=1}^{M} \beta_m \ln k_2^t \qquad (5) \\
&+ \frac{1}{2}\sum_{m=1}^{M}\sum_{n=1}^{M} \beta_{mn}\left[\left(\ln y_m^t \ln k_2^t\right) + \left(\ln y_n^t \ln k_2^t\right) + \left(\ln k_2^t\right)^2\right] \\
&+ \sum_{k=1}^{K} n_k \left(\ln k_1^t\right) R + \sum_{m=1}^{M} \mu_m \left(\ln k_2^t\right) R \; .
\end{aligned}
$$

At any combination of input-output bundles, scale elasticity is:

$$
\begin{aligned}
\varepsilon\left(x^t, y^t\right) &= \sum_k \frac{\partial \ln D^t\left(x_k^t, y_m^t\right)}{\partial \ln x_k^t} + \sum_m \frac{\partial \ln D^t\left(x_k^t, y_m^t\right)}{\partial \ln y_m^t} \\
&= \sum_k \varepsilon_{x,k}^t + \sum_m \varepsilon_{y,m}^t = \varepsilon_x^t + \varepsilon_y^t ,
\end{aligned} \qquad (6)
$$

where

$$
\varepsilon_{x,k}^t = \frac{\partial \ln D^t\left(x_k^t, y_m^t\right)}{\partial \ln x_k^t} = a_k + \sum_{l=1}^{K} a_{kl} \ln x_l^t + \sum_{m=1}^{K} \delta_{km} \ln y_m^t + \sum_{k=1}^{K} n_k R \qquad (6a)
$$

and

$$
\varepsilon_{y,m}^t = \frac{\partial \ln D^t\left(x_k^t, y_m^t\right)}{\partial \ln y_m^t} = \beta_m + \sum_{n=1}^{K} \beta_{mn} \ln y_n^t + \sum_{m=1}^{K} \delta_{km} \ln x_k^t + \sum_{m=1}^{M} \mu_m R \; . \qquad (6b)
$$

Using relations (6a) and (6b), equation (5) can be rewritten as:

$$
\begin{aligned}
\ln D^t\left(x_*^t, y_*^t, R\right) &= \ln D^t\left(x^t, y^t, R\right) + \varepsilon_x^t\left(x_*^t, y_*^t, R\right) \ln k_1^t \\
&+ \varepsilon_y^t\left(x_*^t, y_*^t, R\right) \ln k_2^t + \frac{\left(\ln k_1^t\right)^2}{2} \sum_{k=1}^{K}\sum_{l=1}^{M} a_{kl} \\
&+ \left(\ln k_1^t \ln k_2^t\right) \sum_{k=1}^{K}\sum_{m=1}^{M} \delta_{km} + \frac{\left(\ln k_1^t\right)^2}{2} \sum_{m=1}^{M}\sum_{n=1}^{M} \beta_{mn} .
\end{aligned} \qquad (7)
$$

If we now examine a fully technical and scale efficient firm, or a firm operating on the frontier at the MPSS, we derive that $\ln D^t(x^t, y^t, R) = 0$ and $\ln D^t(x^t_*, y^t_*, R) = 0$. Thus, the left-hand side of (7) is zero. In addition, the first term on the right-hand side is zero and the second term is 1 since $D^t(x^t, y^t)$ is homogeneous of degree 1 in y (Färe et al., 1994). In the light of the above, (7) becomes:

$$\begin{aligned} 0 = \varepsilon^t_x\left(x^t, y^t, R\right)\ln k^t_1 + \ln k^t_2 + \frac{\left(\ln k^t_1\right)^2}{2}\sum_{k=1}^{K}\sum_{l=1}^{K} a_{kl} \\ + \left(\ln k^t_1 \ln k^t_2\right)\sum_{k=1}^{K}\sum_{m=1}^{M}\delta_{km} + \frac{\left(\ln k^t_1\right)^2}{2}\sum_{m=1}^{M}\sum_{n=1}^{M}\beta_{mn}. \end{aligned} \tag{8}$$

By virtue of the linear homogeneity of the output distance function we have:

$$\sum_{n=1}^{N}\sum_{m=1}^{M}\beta_{mn} = \sum_{k=1}^{K}\sum_{m=1}^{M}\delta_{km} = 0\ , \tag{9}$$

which, when taken into consideration with (8), we may re-express (6a) as:

$$\varepsilon^t_x\left(x^t_*, y^t_*, R\right) = \varepsilon^t_x\left(x^t, y^t, R\right) + \frac{1}{2}\sum_{k=1}^{K}\sum_{l=1}^{K} a_{kl}\ln k^t_1 + \sum_{k=1}^{K}\sum_{m=1}^{M}\delta_{km}\ln k^t_2\ , \tag{10}$$

and solving with respect to $\ln k^*_1$ we obtain:

$$\ln k^*_1 = \frac{1-\varepsilon^{i,t}_x}{\Delta} \text{ where } \Delta = \frac{1}{2}\sum_{k=1}^{K}\sum_{l=1}^{K} a_{kl}\ . \tag{10a}$$

We can now estimate $\ln k^t_2$ using (8) and (9):

$$\ln k^t_2 = -\left[\frac{\left(1-\varepsilon^t_x\right)\varepsilon^t_x + \left(1-\varepsilon^t_x\right)^2}{\Delta}\right] = \frac{\varepsilon^t_x - 1}{\Delta}\ . \tag{11}$$

Therefore,

$$\ln k^t_1 - \ln k^t_2 = \frac{\left(1-\varepsilon^t_x\right)}{\Delta} - \frac{\left(\varepsilon^t_x - 1\right)}{\Delta} \Leftrightarrow \frac{k^t_1}{k^t_2} = e^{\frac{2\left[1-(\varepsilon^t_x)\right]}{\Delta}} \tag{12}$$

Recalling relationship (2), it is evident that scale efficiency is:

$$SE\left(x^t\right) = \frac{RAP\left(x^t\right)}{RAP\left(x^t_*\right)} = \frac{k^t_1}{k^t_2} = e^{\frac{2\left[1-(\varepsilon^t_x)\right]}{\Delta}}\ . \tag{13a}$$

It is worth pointing out that, according to (13), $SE \geq 0$ in any case, and $SE = 1$ (that is, the firm is fully scale efficient) only when $\varepsilon_x^t = 1$, which corresponds to a CRS technology. Completely analogously, scale efficiency in period $t+1$ is:

$$SE\left(x^{t+1}\right) = \frac{RAP\left(x^{t+1}\right)}{RAP\left(x_*^{t+1}\right)} = \frac{k_1^{t+1}}{k_2^{t+1}} = e^{\frac{2\left[1-\left(\varepsilon_x^{t+1}\right)\right]}{\Delta}}. \tag{13b}$$

Thus, scale efficiency change between periods t and $t+1$ is easily calculated using relationships (3), (13a), and (13b):

$$SEC\left(x^{t+1}, x^t, \bar{y}\right) = \frac{SE\left(x^{t+1}, \bar{y}\right)}{SE\left(x^t, \bar{y}\right)} = e^{\left[\frac{2\left(\varepsilon_x^{t+1} - \varepsilon_x^t\right)}{\Delta}\right]}. \tag{14}$$

Thus, the output-oriented scale efficiency change depends on the difference of the distance elasticity with respect to inputs between two successive periods, weighted with the sum of coefficients that determine, in some extent, the concavity of the distance function. That is, the scale efficiency change depends not only on the firm's actions but also on the curvature of the frontier, which indicates the position of the MPSS. According to (14) we identify scale efficiency improvement between periods t and $t+1$ if the elasticity of the output distance function with respect to inputs, in absolute terms, is time decreasing and scale efficiency deterioration if the same elasticity increases from period to period.

4. Conclusions

We develop a measure of scale efficiency for the case of an output-oriented, multi-input, multi-output translog distance function using MPSS as a reference point and avoiding the linearity of Balk's (2001) cone technology. Based on this scale efficiency measure, we define an easily calculated measure of scale efficiency change, which may be useful in productivity analysis that employs the Malmquist index.

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