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General Equilibrium Stock Index Futures Pricing Allowing for Event Risk

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Abstract

This study develops a new futures pricing model and derives its analytic solution. Comparative static and simulation results are also presented. Under this general equilibrium framework, we find that bounded degrees of state variables in the broad economy determine co-varying extents among various important market variables. However, increasing event risk, including the sizes of occurrence probability and corresponding impulse effects, makes their analysis intractable.

Key words: general equilibrium model; event risk; intertemporal futures pricing

JEL classification: D52; G13

1. Introduction

Major events like technological innovations and catastrophes often trigger abrupt changes in financial markets. A well known example of the latter is the September 11, 2001, terrorist attacks in the US. The connection between cash and derivative markets in such events becomes an important issue. This study develops a stock index futures pricing model and derives an analytic solution to advance understanding of this connection.

In the literature, asset pricing can be approached by constructing a no-arbitrage portfolio or using a general equilibrium model. The argument of no-arbitrage replication is one of the most exciting ideas in modern finance. Its implications continue to flourish in asset pricing fields. Its axiom is based on a comparison of expected returns and risks from different positions. In particular, by adjusting weights and replicating components repeatedly, an investor may create portfolios without bearing any risk from cash markets. Thus, these portfolios should yield

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risk-free returns. If not, gradual elimination of arbitrage opportunities will achieve this goal eventually. Several studies use this methodology successfully. The most notable is the Black and Scholes (1973) option pricing theory.

In order to make comparisons, the no-arbitrage argument needs traded or marketable assets to replicate positions and adjust weights. This suggests an analytical framework within a partial equilibrium economy because we need exogenous economic variables to assist in asset pricing. It is very useful if traded assets are available in markets; however, some risks may not be simply hedged or otherwise offset because they are non-traded economic variables. An additional assumption is needed to deal with this situation in incomplete markets. For example, Merton (1976) and Cox and Ross (1976) assume a constant Sharpe ratio and diversifiable property to deal with jump risk. The cost of these "unseen" assumptions is that risk premia will appear in fundamental valuation equations and resulting contingent prices (e.g., Cox et al., 1985, henceforth CIR; Hull and White, 1987). These limitations become obstacles for practical implementation of these pricing models.

Second, even though assets or positions for constructing arbitrage portfolios in partial equilibrium models are available, endogenous variables like interest rates and spot prices are affected by salient events simultaneously. Therefore, it is important to realize endogenous properties of these variables and their sensitivity to state variables or major events. A good general equilibrium model helps to provide an exhaustive analytical framework connecting and explaining relationships among economic variables.

A general equilibrium model helps to comprehend various economic phenomena. But one cost of this elegant analytical framework is an unobservable preference assumption of representative agents. Hemler and Longstaff (1991) try to overcome this; in their general equilibrium stock index futures pricing model, an important and surprising feature is that the futures price does not depend on an agent's preference except through a time-preference parameter. This results because the market prices of risk in Equation (31) of CIR are cancelled out by the covariance terms owing to Hemler and Longstaff's (1991) "separable" property. In fact, Hemler and Longstaff's futures pricing formula is a special case of the CIR model.

Furthermore, Hemler and Longstaff (1991) transform state variables from unobservable ones to observable interest rates and spot market volatility. Spot market volatility is rarely discussed in the early futures pricing literature. Under the cost-of-carry model, unbiased expectation theory, or the complete market assumption, partial equilibrium models take spot volatility as a perturbation irrelevant for futures pricing because futures prices reflect discounted expectations of future cash prices. The size of the second moment does not affect the size of the first moment. Hemler and Longstaff (1991) propose a different viewpoint. They use a general equilibrium framework to show that spot volatility does matter in futures pricing because all market variables are endogenous. Thus, volatility in cash markets emerges in their formula. This motivates us to explore a different volatility structure in cash markets: namely the event or jump risk noted in Merton (1976).

The traditional log-normal diffusion process proposed by Black and Scholes (1973) has been widely used in intertemporal asset pricing models:

$$\frac{dS}{S} = \mu dt + \sigma dz \,, \tag{1}$$

where S denotes the return of an underlying asset, μ and σ are its instantaneous drift and diffusion terms, dt is an instantaneous time span, and dz is Brownian motion. Equation (1) means that a percentage change in the spot price S is composed of a certain drift term μdt and a normally distributed stochastic term σdz. As dt gets small, this process predicts that S_{t+dt} will not differ much from S_t. Nevertheless, numerous academic studies found evidence of leptokurtosis, fat tails, and other phenomena deviating from a normal distribution for underlying asset price changes. See for instance Epps and Epps (1976), Kon (1984), and Johnson and Shanno (1987).

The reason for this behavior is explained by Merton (1976) and Cox and Ross (1976): unanticipated jumps during various price evolutions. They assert that surprises or jumps do arrive even in the short run, which make intertemporal models inapplicable or unrealistic when forming dynamic portfolio strategies. Thus, Cox and Ross (1976) employ the following alternative formulation:

$$\frac{dS}{S} = \mu dt + (L-1)d\pi \,, \tag{2}$$

where π is a continuous-time Poisson process, λ is the intensity of the process, and $L-1$ is the jump amplitude. That is, the percent price change is $\mu dt + (L-1)$ with probability λdt and μdt with probability $1-\lambda dt$. Cox and Ross (1976) explain λ as the arrival possibility of an information packet. This means that S changes deterministically via its drift term until a unit of information arrives and an extra stochastic perturbation occurs.

Likewise, Merton (1976) explains these jumps as important pieces of information arriving, which cause extra price changes above and beyond the original Brownian motion. The return dynamics of an underlying asset is written as:

$$\frac{dS}{S} = (\mu - \lambda k)dt + \sigma dz + dq \,, \tag{3}$$

where μ is an instantaneous expected return on the stock allowing for expected jump effect, σ^2 is the instantaneous variance of the return, conditional on no arrivals of important new information, dz is a standard Wiener process, $q(t)$ is a Poisson process which is $L-1$ with probability λdt and 0 with probability $1-\lambda dt$, dz and $q(t)$ are assumed independent, λ is the mean number of new information packet arrivals per unit time, and $k \equiv E(L-1)$, where $L-1$ is the random variable percent change in the asset price if the Poisson event occurs, i.e., an impulse function producing a finite jump in S to $S \times L$.

Assumptions that trading takes place continuously in time and that the underlying stochastic variables follow diffusion-type motions are standard for intertemporal asset pricing models even though they are not completely consistent with reality. The existence of jump risk makes this portfolio hedging risky and attempts to obtain a "no-arbitrage" closed-form solution vain. Cox and Ross (1976) and Merton (1976) solve this problem and derive new option pricing formulas by allowing for occasional jumps. While their original intention is to remedy the problem of replicating discontinuity, their design sheds lights on information-time modeling, while Merton's (1976) design contributes much to valuations of real options, technology innovations, and catastrophe impacts. See Chang et al. (1998) for discussion about the information-time setting, Subramanian (2004) for real options, and Liu et al. (2003) for event risk applications.

There are at least two forerunners of general equilibrium futures pricing models in the literature. Richard and Sundaresan (1981) construct an intertemporal rational expectations model in a multi-good economy with identical consumers. They find that effectiveness of consumption hedging is the key to discerning normal backwardation or the Contango phenomenon. If an investor can successfully use contracts to hedge consumption risks, then the Contango phenomenon prevails. Otherwise, normal backwardation emerges. Cox et al. (1981) derive another closed-form futures pricing formula based on 11 propositions. They show that if a futures price and a bond price are positively correlated, then the futures price is less than the forward price; if they are negatively correlated, then the reverse holds. However, neither pricing formula is preference-free. There are other well-known related studies. For instances, Ramaswamy and Sundaresan (1985) derive values of American options on futures contracts. Their futures pricing formula is not preference-free either. Even though market prices of risk do not enter into their model, the local expectation hypothesis is assumed to hold.

Based on Hemler and Longstaff's (1991) model, our study derives a general equilibrium futures pricing formula to address event risk as noted by Merton (1976). The remainder of this article is organized as follows. We develop an economic model allowing for possible jumps and derive a closed-form pricing formula allowing for event risk in Section 2. Comparative static and simulation results are provided in Section 3. Section 4 concludes.

2. General Equilibrium Stock Index Futures Pricing Allowing for Event Risk

In the world created by Hemler and Longstaff (1991), a fixed number of identical agents seek to maximize their time-additive preferences in a perfectly competitive, continuous economy for risk-free borrowing and lending and for a variety of contingent claims including stock futures contracts. An agent's lifetime utility is of standard log-form. Investment and consumption are described by a single physical good which may be allocated to consumption or to investment. All values are expressed in terms of units of this good with constant returns to scale. Such settings are common in the economic literature. See the Appendix for more technical pricing details.

Production possibility in this economy is not only affected by its growth and volatility features but also by rare jump events. Specifically, it can be described by the following stochastic differential equation with a drift term μX that does not allow for expected jump effects and a diffusion term $\sigma\sqrt{Y}$. These two terms separately represent the growth and volatility features of the production possibility set:

$$\frac{dp}{p} = \mu X dt + \sigma\sqrt{Y}\,dz_P + dq\,. \tag{4}$$

As in Black and Scholes (1973), dz_P is a Wiener process, while as in Merton (1976), dq denotes a Poisson variable governing jump events with probability λdt and $L-1$ is its impulse function. Here two state variables X and Y are designated to represent economic variables. The state variables induce random changes and can be described by the stochastic differential equations:

$$dX = a(b - X)dt + c\sqrt{X}\,dz_X \tag{5}$$

$$dY = f(g - Y)dt + h\sqrt{Y}\,dz_Y\,, \tag{6}$$

where $a, f, c, h > 0$ and dz_X and dz_Y are Wiener processes. Equations (5) and (6) are typical Ornstein-Uhlenback processes where (a, f) are measures of mean-reverting speed, (b, g) are long-term averages, and (c, h) are diffusion terms of (X, Y). Note that we also assume that the jump risk and the Wiener processes dz_X and dz_Y are mutually independent.

A representative investor plans in advance to allocate his consumption C and reinvest his unconsumed wealth in physical production in order to maximize his lifetime utility. In this general equilibrium model, the uniqueness assumption of the production process P guarantees that the agent's wealth W is equivalent to the value of the stock market or the stock index. Note that futures prices are available now and that ρ has the intuitive interpretation of a dividend yield. Nevertheless, X and Y are unobservable state variables, and we transform variables in order to more clearly express the futures price.

From standard stochastic programming procedures, we obtain the equilibrium interest rate:

$$r = (\mu X + \lambda k) - V\,. \tag{7}$$

That is, the equilibrium riskless interest rate equals the expected return of the production activity μX plus the expected jump effect λk minus the variance of market returns V, which completes our variable transformation from the unobservable state variables X and Y to the observable interest rate r and the market variance V. We can now obtain the dynamic of the value of the stock market in terms of observable r and V:

$$dW = W\left(r + V - \lambda k - \rho\right)dt + W\sqrt{V}dz_P + Wdq\ . \tag{8}$$

According to the general equilibrium framework of Hemler and Longstaff (1991) and Cox et al. (1985), the additional jump risk differentiates the equilibrium interest rate with the expected impulse component $-\lambda k$. Thus, the stock index futures price F satisfies the fundamental valuation equation:

$$\begin{aligned} &F_W W\left(r - \rho\right) + F_V\left[f\left(\alpha - V\right)\right] + F_r\left[a\left(\varepsilon + \lambda k - \varphi V - r\right)\right] \\ &\quad + \frac{1}{2}F_{WW}W^2V + \frac{1}{2}F_{VV}\gamma^2V + \frac{1}{2}F_{rr}\left[\phi^2\left(r + V - \lambda k\right) + \eta^2V\right] \\ &\quad + F_{Vr}\gamma\eta V = F_\tau \end{aligned} \tag{9}$$

subject to $F(W, r, V, q, 0) = W(T)$. According to Equation (9), it is obvious that other than the time-preference parameter ρ, futures prices do not depend on any preference parameter after allowing for jump effects. On the other hand, although the futures price is explicitly a function of non-traded variables, the covariance terms of index returns with changes in r and V affect the expected returns and the variances of futures dynamics simultaneously. These two terms exactly offset each other.

Since under the independence assumption that the jump risk is uncorrelated with dz_P, dz_X, and dz_Y, expectation operation allowing for jump randomness is separable. We can solve (9) and obtain the equilibrium futures price:

$$F(W, r, V, q, \tau) = We^{-\rho\tau}Q_1\left(\tau\right)e^{Q_2(\tau)+Q_3(\tau)r+Q_4(\tau)V}\ , \tag{10}$$

where

$$Q_1\left(\tau\right) = \left[\frac{2\kappa e^{\frac{1}{2}(a+\kappa)\tau}}{\left(a+\kappa\right)\left(e^{\kappa\tau}-1\right)+2\kappa}\right]^{-2\frac{f\alpha + a\left(\varepsilon+\lambda k\right)}{\phi^2}} \times \left[\frac{2\nu e^{\frac{1}{2}(f+\nu)\tau}}{\left(e^{\nu\tau}-1\right)\left(f+\nu\right)+2\nu}\right]^{2\frac{f\alpha}{\gamma^2}}$$

$$\times\left[\frac{2\kappa}{\left(e^{\kappa\tau}-1\right)\left(a+\kappa\right)+2\kappa}\right]^{-2\frac{a}{\phi^2}\lambda k}$$

$$Q_2\left(\tau\right) = \frac{\lambda k}{2\phi^2}\left[\frac{4\left(a-\kappa\right)\kappa}{\left(e^{\kappa\tau}-1\right)\left(a+\kappa\right)+2\kappa} - \left(a+\kappa\right)^2\tau - 2a + 2\kappa\right]$$

$$Q_3\left(\tau\right) = \frac{2\left(e^{\kappa\tau}-1\right)}{\left(a+\kappa\right)\left(e^{\kappa\tau}-1\right)+2\kappa}$$

$$Q_4\left(\tau\right) = Q_3\left(\tau\right) + \frac{2\left(1-e^{\nu\tau}\right)}{\left(\nu+f\right)\left(e^{\nu\tau}-1\right)+2\nu}$$

$$\kappa = \sqrt{a^2 - 2\phi^2}, \quad \nu = \sqrt{f^2 + 2\gamma^2}\ .$$

Equation (10) shows that the equilibrium stock index futures price is a function of W, r, ν, and τ, the dividend-yield/time-preference parameter ρ, the expected impulse effect of the percent change due to the jump event on the production activity k, and the instantaneous jump occurrence probability, λ. Substituting $\tau = 0$ into (10) verifies that the equilibrium stock index futures price satisfies the boundary condition $F(W, r, V, q, 0) = W(T)$ at expiration date.

Actually, (10) is not complete without specifying the expected impulse effect of the jump event. Merton (1976) discusses two possible situations. If his "special case 1" assumption is adopted, then $k = -1$ and the instantaneous equilibrium interest rate is downsized by λ. This is also described by Samuelson (1973): there is a positive probability of immediate ruin. If "special case 2" is valid, then the random variable L has a log-normal distribution with mean k, and the instantaneous equilibrium interest rate is marked up by λk. If we ignore the jump effect (i.e., if $\lambda = 0$), then (10) is simplifies to Hemler and Longstaff's (1991) case.

3. Results of Comparative Static and Simulation Analysis

3.1 Comparative Statics of Equilibrium Futures Prices and Variables

Comparative static results help us to understand relationships between the general equilibrium futures price and other important market variables. From (10), it is evident that:

$$\frac{\partial F}{\partial W} = \frac{F}{W} = e^{-\rho\tau} Q_1(\tau) e^{Q_2(\tau)+Q_3(\tau)r+Q_4(\tau)V} \tag{11}$$

$$\frac{\partial F}{\partial \rho} = -\tau \times F \tag{12}$$

$$\frac{\partial F}{\partial r} = \frac{2\left(e^{\kappa\tau}-1\right)}{(a+\kappa)\left(e^{\kappa\tau}-1\right)+2\kappa} \times F \tag{13}$$

$$\frac{\partial F}{\partial V} = \left[\frac{2\left(e^{\kappa\tau}-1\right)}{(a+\kappa)\left(e^{\kappa\tau}-1\right)+2\kappa} - \frac{2\left(e^{\nu\tau}-1\right)}{(\nu+f)\left(e^{\nu\tau}-1\right)+2\nu}\right] \times F \tag{14}$$

$$\begin{aligned}\frac{\partial F}{\partial \lambda} = \frac{k}{\phi^2}\Bigg\{ & \frac{2(a-\kappa)\kappa}{\left(e^{\kappa\tau}-1\right)(a+\kappa)+2\kappa} - \left[\left(2a^2-\phi^2+2a\kappa\right)\tau + (a-\kappa)\right] \\ & -4a\ln\left[\frac{2\kappa}{\left(e^{\kappa\tau}-1\right)(a+\kappa)+2\kappa}\right]\Bigg\} F\end{aligned} \tag{15}$$

$$\begin{aligned}\frac{\partial F}{\partial k} = \frac{\lambda}{\phi^2}\Bigg\{ & \frac{2(a-\kappa)\kappa}{\left(e^{\kappa\tau}-1\right)(a+\kappa)+2\kappa} - \left[\left(2a^2-\phi^2+2a\kappa\right)\tau + (a-\kappa)\right] \\ & -4a\ln\left[\frac{2\kappa}{\left(e^{\kappa\tau}-1\right)(a+\kappa)+2\kappa}\right]\Bigg\} F\end{aligned} \tag{16}$$

$$
\begin{aligned}
\frac{\partial F}{\partial \tau} = {} & \frac{r\kappa^2}{\left[\kappa\cosh\left(\frac{\kappa\tau}{2}\right)+a\sinh\left(\frac{\kappa\tau}{2}\right)\right]^2} \\
& -\lambda k\left[\frac{(a+\kappa)^2}{2\phi^2}+\frac{a^2\kappa^2+\kappa^4}{2\phi^2\left[\kappa\cosh\left(\frac{\kappa\tau}{2}\right)+a\sinh\left(\frac{\kappa\tau}{2}\right)\right]^2}\right] \\
& +\frac{4e^{\frac{1}{2}(\kappa+\nu)\tau}\gamma^2(a+\kappa)\nu\cosh\left(\frac{\nu\tau}{2}\right)}{\phi^2\gamma^2\left[-a+\kappa+e^{\kappa\tau}(a+\kappa)\right]\left[-f+\nu+e^{\nu\tau}(f+\nu)\right]} \\
& \left\{\lambda ka\kappa\cosh\left(\frac{\nu\tau}{2}\right)-\left[(f\alpha+a\varepsilon)(a-\kappa)+\lambda ka(a-2\kappa)\right]\sinh\left(\frac{\kappa\tau}{2}\right)\right. \\
& +f\kappa\gamma^2\left[\lambda ka(a+\kappa)+2\alpha\phi^2\right]\cosh\left(\frac{\kappa\tau}{2}\right) \\
& -\left[2f\gamma^2\left[\phi^2(f\alpha+a\varepsilon)+\lambda ka(a+\kappa)(a-2\kappa)\right]-\alpha\alpha\phi^2\gamma^2\right] \\
& \left.\times\sinh\left(\frac{\kappa\tau}{2}\right)\sinh\left(\frac{\nu\tau}{2}\right)\right\} \\
& +\frac{\kappa^2(f^2+\nu^2)\cosh(\nu\tau)+2\kappa\nu\left[-a\nu\sinh(\kappa\tau)+f\kappa\sinh(\nu\tau)\right]}{\left[-a+\kappa+e^{\kappa\tau}(a+\kappa)\right]^2\left[-f+\nu+e^{\nu\tau}(f+\nu)\right]^2}
\end{aligned}
\tag{17}
$$

where $\cosh(\cdot)=(e^{(\cdot)}+e^{-(\cdot)})/2$ and $\sinh(\cdot)=(e^{(\cdot)}-e^{-(\cdot)})/2$.

It is worthwhile to note several functional regularities and behavioral assumptions before proceeding. First, absolute values of negative mean-reverting parameters represent contraction to long-term averages. That is, the dynamics of the interest rate r and the spot volatility V become more stable or more tightly bounded with larger a and f. Smaller diffusion terms in the dynamics of r and V create the same effect. In consequence, smaller γ^2, smaller ϕ^2, and larger κ are equivalent. Second, $Q_1(\tau)$ in (10) must be positive to accord with the non-negative property of the stock index W. This condition implies that $a^2 > 2\phi^2$ in functional regularity and that the interest rate dynamics are stable, in agreement with economic intuition. This condition also guarantees that $Q_3(\tau)$ is positive. Nevertheless, $Q_2(\tau)$ and $Q_4(\tau)$ are indeterminate. Third, λ is nonnegative because it is an occurrence probability and k is undetermined and depends on properties of jump events. For instance, k is positive for new major technology innovations but negative for catastrophes.

Equation (11) shows that the general equilibrium stock index futures price F is a positive and monotonically increasing function of the stock index level W. This is due to the nonnegative dynamics of the square roots of $Q_1(\tau)$ and W. Similarly, the stock index futures price is a decreasing function of the time preference parameter ρ as shown in (12). Next, Equation (13) reveals that the stock index futures price is a uniformly increasing function of the risk-free rate if the functional regularity $a^2 > 2\phi^2$ holds. This result is consistent with the cost-of-carry model that the interest rate is a carrying cost for underlying assets. In addition, this relationship becomes even stronger with a longer time-to-maturity τ and a smaller mean-reverting tendency κ, i.e., within a more stable economy.

The partial differentiation result in (14) shows that the relationship between the stock index futures price and the spot market volatility is indeterminate because $2(e^{\kappa\tau}-1)/(a+\kappa)(e^{\kappa\tau}-1)+2\kappa > 0$ while $-2(e^{\nu\tau}-1)/(\nu+f)(e^{\nu\tau}-1)+2\nu < 0$. It is interesting to notice that we can view the former roughly as a bounded measure of the dynamics of the interest rate since a larger κ or a larger a makes $2(e^{\kappa\tau}-1)/(a+\kappa)(e^{\kappa\tau}-1)+2\kappa$ smaller, and thus the bounded interest rate dynamic tends to yield a negative relationship between the futures price and the spot market volatility. In contrast, a more tightly bounded spot volatility dynamic comes from a larger f or a smaller γ; however, they affect $2(e^{\nu\tau}-1)/(\nu+f)(e^{\nu\tau}-1)+2\nu$ in opposite directions. Therefore, the total effect from the volatility dynamic in the spot market is inconclusive.

Even though the partial differential results in (15) and (16) show indeterminate comparative static results about the jump occurrence probability λ and the corresponding expected impulse effect k, it is obvious that connections between possible jump events and the equilibrium futures prices are related with the bounded degrees of the dynamics of the interest rate and spot volatility. The relationship between the length of time-to-maturity τ and the equilibrium stock index futures price shown in (17) is difficult to comprehend. The bounded degrees of the interest rate and the spot volatility in the economy and the expected impulse effect interact in a very complicated manner. Thus, convergence toward Contango or backwardation between the equilibrium futures prices and the stock index before the expiration date is not monotonic as the cost-of-carry model predicts. We discuss this issue further in the next section.

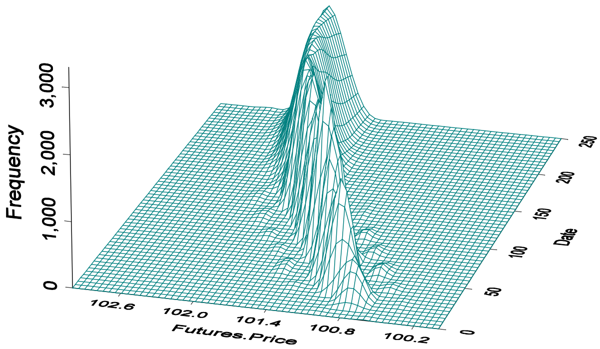
3.2 Simulations of Equilibrium Futures Prices and Variables

From the results of the comparative statics, we know that the indeterminate relationships are related with the bounded degrees of variables and the sizes of jump parameters. Therefore, in this subsection we simulate different scenarios categorized by bounded degrees of the economy and the occurrence of jump events to visualize relationships among important variables. In each simulation scenario, $251 \times 5{,}000$ trials are generated to simulate variable trajectories and distributions. Results are presented accordingly in Figure 1.

The three time-series distributions shown in the figure convey clear information that stationary conditions of the economy to determine dispersed degrees of futures price evolving dynamics. With larger diffusion terms σ, c, and h, and smaller mean-reverting speed coefficients a and f, the peak of Panel B is fatter than in Panel A. Moreover, disconnected jump events exaggerate such uncertainty in an unpredictable way as irregular emerging spikes in Panel C. That is, jumps in the index lead to distributions for spot and futures prices that have degrees of skewness and kurtosis that are very different from those distributions that disallow for event risk.

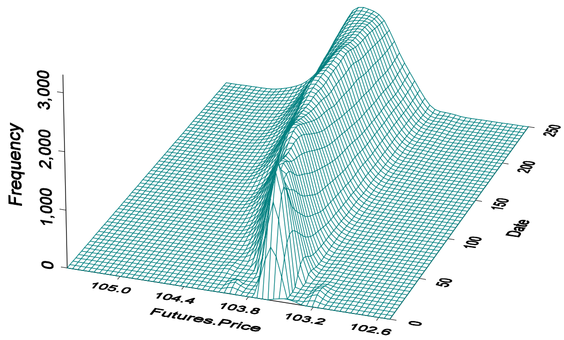
Figure 1. Simulated Futures Price Time-Series Distributions within Different Bounded Degrees of Economy with- and without Allowing Jump Effects

(A) Simulated futures price time-series distribution within a bounded economy without jump effects



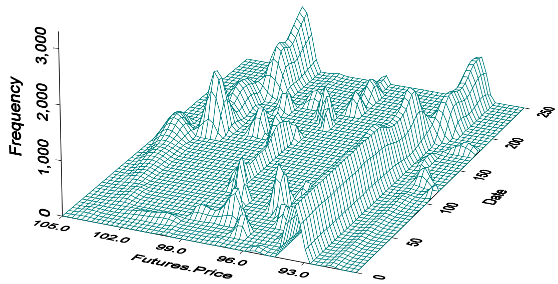
Parameter settings: $\mu = 0.02$, $\sigma = 0.02$, $a = 0.95$, $b = 1$, $c = 0.02$, $f = 0.95$, $g = 1$, $h = 0.02$, $\rho = 0$, $T - t = 250$, and $\lambda = 0$.

(B) Simulated futures price time-series distribution within a less bounded economy without jump effects



Parameter settings: $\mu = 0.04$, $\sigma = 0.04$, $a = 0.05$, $b = 1$, $c = 0.04$, $f = 0.05$, $g = 1$, $h = 0.04$, $\rho = 0$, $T - t = 250$, and $\lambda = 0$.

(C) Simulated futures price time-series distribution within a less bounded economy with jump effects



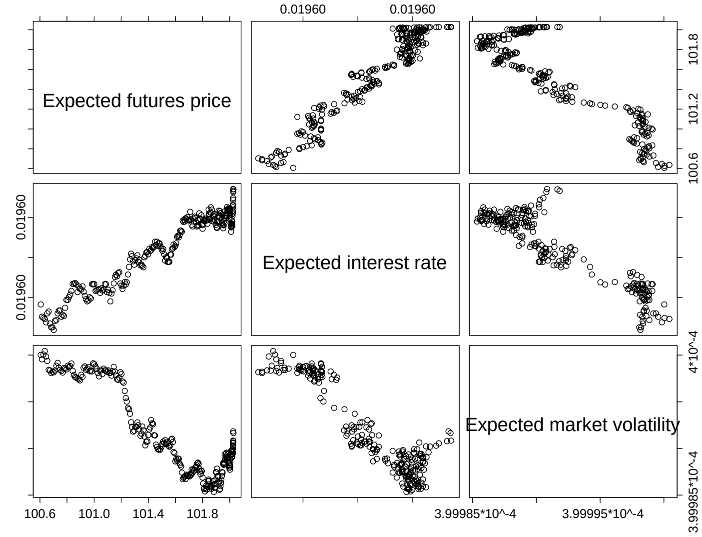
Parameter settings: $\mu = 0.04$, $\sigma = 0.04$, $a = 0.05$, $b = 1$, $c = 0.04$, $f = 0.05$, $g = 1$, $h = 0.04$, $\lambda = 0.01$, $\rho = 0$, $T - t = 250$, and $k = 0$ (with corresponding impulse effects $L - 1 = \pm 10\%$ with equal occurring probability 0.5).

It is worthwhile to examine relationships among the futures price, interest rate, and spot volatility by simulations as well. In a bounded economy without event risk as shown in Panel A of Figure 2, the three variables interact with each other in a clear manner as simulation points scatter adjacently within very narrow bands. In contrast, these bands get wider in a less bounded economy without jump events as shown in Panel B. Nonetheless, they are still correlated with the same signs as in Panel A. For instance, the expected futures price is positively correlated with the expected interest rate, and the expected interest rate is negatively correlated with the spot volatility. However, Panel C exhibits wildly different, more complex relationships after the introduction of event risk. This explains why the literature separates possible jumps from spot volatility when discussing various issues even when the stochastic spot volatility setting is also taken into account. See Duffie et al. (2000) and Liu et al. (2003) for discussion.

The cost-of-carry model claims that if carrying cost is positive, the futures price is greater than the spot price and the Contango phenomenon prevails. In contrast, backwardation occurs when the futures price is less than the spot price or when carrying cost is negative. In our case, the carrying cost is $r - \rho$, and the converging tendency between spot and futures prices is determined by exogenous parameters, cash market volatility V, and the endogenous interest rate r jointly as shown in (10). Although without a definite monotonic pattern, it is obvious that the cash market volatility V and the interest rate r are key to determining Contango or backwardation and the exogenous combination of those parameters. Since r varies inversely with V as shown in Panels A and B, we predict that an economy with a greater unbounded degree or a higher level of market volatility has a higher probability of backwardation. Nonetheless, it is not always the case that an higher level of jump effects makes the relationships as complex as shown in Panel C.

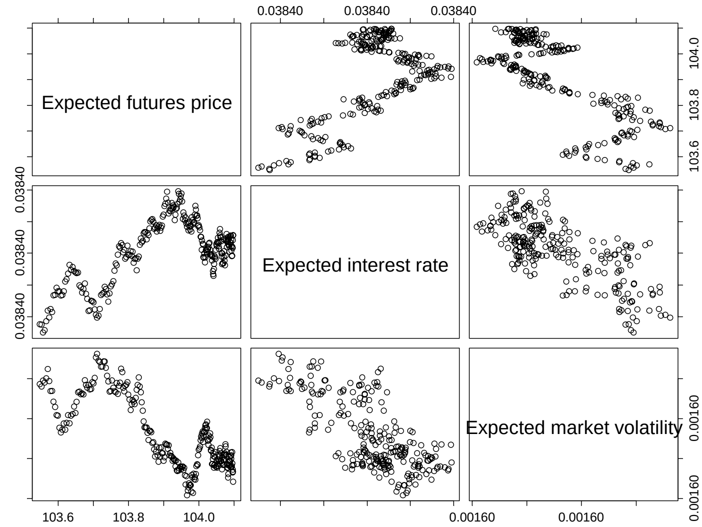
Figure 2. Simulated Relationships among Expected Futures Price, Expected Interest Rate, and Expected Spot Market Volatility

(A) Bounded economy without jump effects



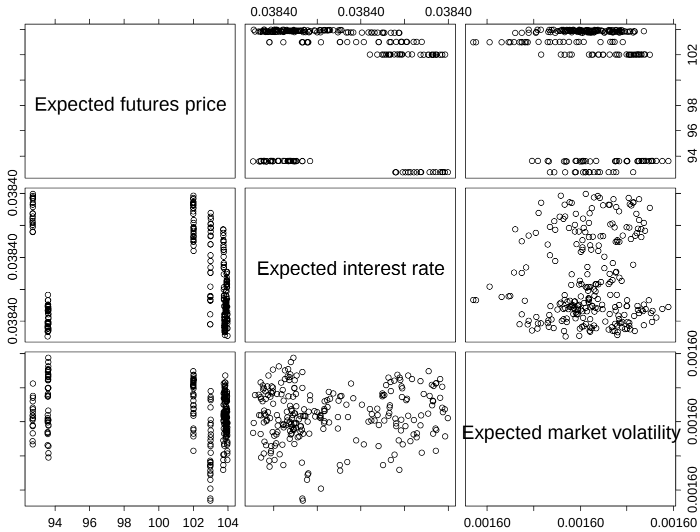
Parameter settings: $\mu = 0.02$, $\sigma = 0.02$, $a = 0.95$, $b = 1$, $c = 0.02$, $f = 0.95$, $g = 1$, $h = 0.02$, $\rho = 0$, $T - t = 250$, and $\lambda = 0$.

(B) Less bounded economy without jump effects



Parameter settings: $\mu = 0.04$, $\sigma = 0.04$, $a = 0.05$, $b = 1$, $c = 0.04$, $f = 0.05$, $g = 1$, $h = 0.04$, $\rho = 0$, $T - t = 250$, and $\lambda = 0$.

(C) Less bounded economy with jump effects



Parameter settings: $\mu = 0.04$, $\sigma = 0.04$, $a = 0.05$, $b = 1$, $c = 0.04$, $f = 0.05$, $g = 1$, $h = 0.04$, $\lambda = 0.01$, $\rho = 0$, $T - t = 250$, and $k = 0$ (with corresponding impulse effects $L - 1 = \pm 10\%$ with equal occurring probability 0.5).

4. Concluding Remarks

Numerous studies have constructed models that explicitly allow for large market movements or fat tails in return distributions. For example, Merton (1976) notes that the Black-Scholes continuous-time framework can be improved upon with a mixture of a diffusion process and a Poisson-directed process. The diffusion process component can be used to represent the frequent local changes and the Poisson-directed component can be used to incorporate rare but influential events such as major technology innovations or catastrophes. Hemler and Longstaff (1991) propose a general equilibrium model that addresses the role of stochastic volatility in pricing futures contracts.

By adapting Hemler and Longstaff's (1991) preference-free model and Merton's (1976) jump setting, this study develops a new futures pricing model that differs considerably from the usual cost-of-carry model. According to our closed-form solution and comparative statics and simulation results, we find that market volatility and jump events, which are not shown in the cost-of-carry model, affect futures pricing due to dynamic and endogenous connections. We find that a decreasing level of cash market volatility tends to create a Contango convergence pattern in that it varies inversely with the equilibrium interest rate. Nevertheless, the emergency of jump events makes relationships among economic variables indeterminate.

Appendix

We assume that identical agents seek to maximize their time-additive preferences. Their lifetime utility is of the form:

$$E_t\left[\int_t^{\infty} e^{-\rho s}\ln(C(s))ds\right], \tag{A1}$$

where $E_t[\cdot]$ is a conditional expectation operator, $C(s)$ denotes consumption at time s, and ρ is the agent's intertemporal discount rate of lifetime utility. Since the underlying asset is a stock index, we assume that the economy can be described by a single physical good that may be allocated to consumption or investment. It is governed by the following stochastic differential equation with a drift term μX that does not allow for expected jump effects and a diffusion term $\sigma\sqrt{Y}$:

$$\frac{dp}{p} = \mu X dt + \sigma\sqrt{Y}dz_P + dq \ , \tag{A2}$$

where dz_P is a Wiener process and $dq = dn(t)\times(L-1)$, with $dn(t)$ a Poisson variable controlling jump occurrence with probability λdt and $k \equiv L-1$ is a impulse function of the percent change in production activity due to the jump event. Here X and Y are economic state variables that induce random changes and can be described by the stochastic differential equations:

$$dX = a(b-X)dt + c\sqrt{X}\,dz_X \tag{A3}$$

$$dY = f(g-Y)dt + h\sqrt{Y}dz_Y \ , \tag{A4}$$

where $a, f, c, h > 0$ and dz_X and dz_Y are Wiener processes. Equations (A3) and (A4) are typical Ornstein-Uhlenback processes, where (a, f) are measures of mean-reverting speed, (b, g) are long-term averages, and (c, h) are diffusion terms of (X, Y). Note that we also assume that the jump risk and the Wiener processes dz_X and dz_Y are mutually independent.

A representative investor plans in advance how she will allocate her consumption C and investing unconsumed wealth in physical production in order to maximize lifetime utility, (A1), subject to the budget constraint:

$$dW = W\frac{dp}{p} - Cdt \ , \tag{A5}$$

where W is the agent's wealth. Thus, the value function can be stated as:

$$J(W, X, Y, q, t) = \frac{e^{-\rho t}}{\rho}\ln(W) + G(X, Y, q, t), \tag{A6}$$

and the optimal consumption is ρW. Substituting the optimal consumption into

(A5) gives the wealth dynamics:

$$\frac{dW}{W} = (\mu X - \rho)dt + \sigma\sqrt{Y}dz_P + dq \ . \tag{A7}$$

In this general equilibrium model, the uniqueness assumption of the production process P guarantees that the wealth W is equivalent to the value of the stock market and (A7) can be used to represent the stock index dynamics. Nevertheless, X and Y are unobservable state variables, and we transform variables in order to more clearly express the futures price.

From standard stochastic programming procedures, we can obtain the equilibrium interest rate:

$$r = (\mu X + \lambda k) - V \ . \tag{A8}$$

That is, the equilibrium riskless interest rate equals the expected return on the production activity μX plus the expected jump effect λk minus the variance of market return V, where:

$$V = \sigma^2 Y \tag{A9}$$

Equations (A8) and (A9) form a simultaneous linear system that is globally invertible for X and Y. That is:

$$X = \frac{1}{\mu}(r + V - \lambda k) \tag{A10}$$

$$Y = \frac{V}{\sigma^2} \ . \tag{A11}$$

Equations (A10) and (A11) help to complete our variable transformations from the unobservable state variables X and Y to the observable interest rate r and market variance V. Applying Itô's lemma gives the following dynamics of r and V:

$$dV = f(\alpha - V)dt + \gamma\sqrt{V}dz_Y \tag{A12}$$

$$dr = a[(\varepsilon + \lambda k - \varphi V) - r]dt + \phi\sqrt{r + V - \lambda k}\,dz_X + \eta\sqrt{V}dz_Y \ , \tag{A13}$$

where $\alpha = \sigma^2 g$, $\gamma = \sigma h$, $\varepsilon = (\mu b - f\alpha/a)$, $\varphi = (1 - f/a)$, $\phi = c\sqrt{\mu}$, and $\eta = -\gamma$. In addition, we can obtain the dynamic of the value of the stock market in terms of observable r and V:

$$dW = W(r + V - \lambda k - \rho)dt + W\sqrt{V}dz_P + Wdq \ . \tag{A14}$$

According to the general equilibrium framework of Hemler and Longstaff

(1991) and Cox et al. (1985), it is evident that the additional jump risk differentiates the equilibrium interest rate with the extra component $-\lambda k$. Thus, the stock index futures price F satisfies the fundamental valuation equation:

$$
\begin{aligned}
&F_W W(r-\rho)+F_V\left[f(\alpha-V)\right]+F_r\left[a(\varepsilon+\lambda k-\varphi V-r)\right]+\frac{1}{2}F_{WW}W^2V \\
&\quad+\frac{1}{2}F_{VV}\gamma^2V+\frac{1}{2}F_{rr}\left[\phi^2(r+V-\lambda k)+\eta^2V\right]+F_{Vr}\gamma\eta V=F_\tau
\end{aligned}
\tag{A15}
$$

subject to $F(W,r,V,q,0)=W(T)$. We can solve (A15) and obtain the equilibrium futures price:

$$
F(W,r,V,q,\tau)=We^{-\rho\tau}Q_1(\tau)e^{Q_2(\tau)+Q_3(\tau)r+Q_4(\tau)V}, \tag{A16}
$$

where

$$
\begin{aligned}
Q_1(\tau)&=\left[\frac{2\kappa e^{\frac{1}{2}(a+\kappa)\tau}}{(a+\kappa)(e^{\kappa\tau}-1)+2\kappa}\right]^{-2\frac{f\alpha+a(\varepsilon+\lambda k)}{\phi^2}}\times\left[\frac{2\nu e^{\frac{1}{2}(f+\nu)\tau}}{(e^{\nu\tau}-1)(f+\nu)+2\nu}\right]^{2\frac{f\alpha}{\gamma^2}} \\
&\quad\times\left[\frac{2\kappa}{(e^{\kappa\tau}-1)(a+\kappa)+2\kappa}\right]^{-2\frac{a}{\phi^2}\lambda k} \\
Q_2(\tau)&=\frac{\lambda k}{2\phi^2}\left[\frac{4(a-\kappa)\kappa}{(e^{\kappa\tau}-1)(a+\kappa)+2\kappa}-(a+\kappa)^2\tau-2a+2\kappa\right] \\
Q_3(\tau)&=\frac{2(e^{\kappa\tau}-1)}{(a+\kappa)(e^{\kappa\tau}-1)+2\kappa} \\
Q_4(\tau)&=Q_3(\tau)+\frac{2(1-e^{\nu\tau})}{(\nu+f)(e^{\nu\tau}-1)+2\nu} \\
\kappa&=\sqrt{a^2-2\phi^2},\quad \nu=\sqrt{f^2+2\gamma^2}.
\end{aligned}
$$

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