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A Note on Optimality of Full-Capacity Flat-Rate Equilibrium

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1. Introduction

Balasko (2001) provides a type of Arrow-Debreu model with time-differentiated goods and production subject to a capacity constraint. The time differentiated and yet physically identical goods could be electricity, natural gas, or some other kind of public utility. The unit cost of production is the same across periods, although it may be different from the unit cost of capacity creation. In a subsequent paper, Balasko (2008) explores the concept of full-capacity flat-rate equilibrium when (as is generally the case) there is a unique peak period of production. A full-capacity flat-rate equilibrium is a uniform price of the dated commodity such that the quantities demanded by the consumers at that uniform price can be met by the firm after earning zero profits. Rationing peak-demand (in general) is proved to Pareto dominate the non-rationing one.

In this note we try to find out whether the non-optimality results reported in Balasko (2008) continue to be valid after dropping the assumption of constant unit costs of production across periods. Time-varying (or non-stationary) unit costs of production would arise if one was to consider the situation where electricity is generated from solar energy rather than the more conventional sources.

Unlike Balasko (2008), here it is possible that there will be "an exceptional case" where the associated allocation is Pareto optimal. By "an exceptional case" we mean a situation where the flat-rate price equals the unit production cost in any off-peak period which in turn is equal to the unit cost of production during the peak period plus the unit capacity installation cost. We show that a full-capacity flat-rate equilibrium satisfying the unique peak-period property is Pareto optimal if the equilibrium itself is "an exceptional case."

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2. The Model

As in Balasko (2001) there are $T+1$ goods, consisting of T physically identical dated goods, and a numeraire whose price is set to 1. The first T goods are the dated goods, whereas the $(T+1)$ -th good is the numeraire.

Let $\vec{p} = (p_t)$, with $t = 1,\ldots,T$, denote the T -vector whose components are the prices $p_t > 0$ of the physically identical goods.

The physically identical goods are produced by one *socially owned* firm out of the numeraire input. The size of the firm is given by its capacity, which imposes an upper bound on the amount of the dated good that can be produced in any period. If $\vec{y} \in \Re_+^T$ denotes the firm's output vector, the output y^t in period t cannot exceed the installed capacity.

The unit capacity costs and the time-varying unit running costs are denoted respectively $\rho > 0$ and $\gamma_t > 0$ for $t = 1,\ldots,T$. One unit of capacity is built with $\rho > 0$ units of numeraire; the production of one unit of output in period $t \in \{1,\ldots,T\}$ requires γ_t units of numeraire if there is sufficient capacity. Hence the minimal quantity of the numeraire good that is necessary for the production of $\vec{y} \in \Re_+^T$ is $\sum_{t=1}^T \gamma_t y^t + \rho \max_{1\le t\le T} y^t$.

An activity vector is a pair $(\vec{y}, -\lambda) \in \Re_+^T \times (-\Re_+)$, where $\vec{y}$ denotes an output vector and λ is the amount of numeraire used by the firm to produce $\vec{y}$. The activity vector $(\vec{y}, -\lambda)$ is said to be *feasible* if $\lambda \ge \sum_{t=1}^T \gamma_t y^t + \rho \max_{1\le t\le T} y^t$. Let Y denote the set of all feasible activity vectors. Clearly the null (activity) vector in $\Re^{T+1}$ (i.e., the zero vector) belongs to Y . Further, $(\vec{y}, -\lambda) \in Y$ if and only if $(\alpha\vec{y}, -\alpha\lambda) \in Y$ for all $\alpha \ge 0$.

Given the price vector $\vec{p}$ the profit associated with the activity vector $(\vec{y}, -\lambda)$ is equal to $\vec{p} \,.\, \vec{y} - \lambda$.

There is a finite number m of consumers indexed by $i = 1,\ldots,m$. Consumer i has consumption bundle denoted $(x(i), \xi(i)) = (x^1(i), x^2(i),\ldots, x^T(i), \xi(i)) \in \Re_{++}^{T+1}$. Consumer i is initially endowed with the quantity $\omega(i) > 0$ of numeraire.

The consumption space of all consumers is the same (i.e., $\Re_{++}^{T+1}$). The preferences of consumer i are represented by a twice continuously differentiable utility function $u_i : \Re_{++}^{T+1} \to \Re$ that satisfies the following conditions: (i) $Du_i(x(i), \xi(i)) \in \Re_{++}^{T+1}$, (ii) $D^2u_i(x(i), \xi(i))$ is negative definite, and (iii) for all $(x(i), \xi(i)) \in \Re_{++}^{T+1}$ the indifference set through $(x(i), \xi(i))$ (i.e. the set $\{(x'(i), \xi'(i)) \in \Re_{++}^{T+1} : u_i(x'(i), \xi'(i)) = u_i(x(i), \xi(i))\}$) is closed in $\Re_{++}^{T+1}$. Thus u_i is smoothly strictly increasing and smoothly strictly concave.

An *allocation* is an array $[<(x(i), \xi(i)) : i = 1,\ldots,m>, (y, -\lambda)]$, where for $i = 1,\ldots,m$, $(x(i), \xi(i))$ is a consumption bundle and $(y, -\lambda)$ is an activity vector. It is said to be *attainable* if (i) $(y, -\lambda) \in Y$, (ii) $\sum_{i=1}^m x^t(i) \le y^t$ for all $t = 1,\ldots,T$, and (iii) $\sum_{i=1}^m \xi(i) + \lambda = \sum_{i=1}^m \omega(i)$.

An allocation $[<(x'(i), \xi'(i)) : i = 1,\ldots,m>, (y', -\lambda')]$ is said to *Pareto dominate* an allocation $[<(x(i), \xi(i)) : i = 1,\ldots,m>, (y, -\lambda)]$ if (i) the former is attainable, (ii) $u_i(x'(i), \xi'(i)) \ge u_i(x(i), \xi(i))$ for $i = 1,\ldots,m$, and (iii)

$u_i(x'(i), \xi'(i)) > u_i(x(i), \xi(i))$ for some $i \in \{1, \ldots, m\}$.

An attainable allocation is said to be *Pareto optimal* if it is not Pareto dominated by any other allocation.

Given a price vector $\vec{p}$, the consumption vector $(x(i, \vec{p}), \xi(i, \vec{p}))$ maximizes the utility function $u_i(x(i), \xi(i))$ subject to the budget constraint $\vec{p}\,.\,x(i) + \xi(i)$. For a given a price vector $\vec{p}$, let $y(\vec{p}) = \sum_{i=1}^{m} x(i, \vec{p})$, $y^t(\vec{p}) = \sum_{i=1}^{m} x^t(i, \vec{p})$ for $t = 1, \ldots, T$ and $\phi(\vec{p}) = \sum_{i=1}^{m} \xi(i, \vec{p})$.

Let $\vec{\gamma} = (\gamma_t)$, with $t = 1, \ldots, T$, denote the T-vector whose components are as defined earlier (i.e., the unit cost of producing the dated good in period t). The price vector $\vec{p}$ is said to be a *full-capacity equilibrium* if $(\vec{p} - \vec{\gamma}) \cdot y(\vec{p}) - \rho \max_{1 \le t \le T} y^t(\vec{p})$ $= 0$. In other words, $\vec{p}$ is a full-capacity equilibrium if the demand vector $y(\vec{p})$ can be satisfied with zero profit.

Note: If $(\vec{p} - \vec{\gamma}) \cdot y(\vec{p}) - \rho \max_{1 \le t \le T} y^t(\vec{p}) = 0$, then since $\phi(\vec{p}) = \sum_{i=1}^{m} \omega(i) - \vec{p} \cdot y(\vec{p})$, we get that $\vec{\gamma} \cdot y(\vec{p}) + \rho \max_{1 \le t \le T} y^t(\vec{p}) + \phi(\vec{p}) = \sum_{i=1}^{m} \omega(i)$.

Let e denote the T-vector with all coordinates equal to 1. A *flat-rate* price vector $\vec{p}$ is a price vector such that $\vec{p} = pe$ for some positive real number p. A positive real number p is said to be a *full-capacity flat-rate equilibrium* if the price vector $\vec{p} = pe$ is a full-capacity equilibrium.

If p is a full-capacity flat-rate equilibrium, then the allocation $[<(x(i, pe), \xi(i, pe)) : i = 1, \ldots, m>, (y(pe), -\lambda)]$, where $\lambda = \vec{\gamma} \cdot y(pe) + \rho \max_{1 \le t \le T} y^t(pe)$ is said to be the *allocation associated with* p.

Let p be a full-capacity flat-rate equilibrium and let $[<(x(i, pe), \xi(i, pe)) : i = 1, \ldots, m>, (y(pe), -\lambda)]$ be the allocation associated with p. Then p will be said to satisfy the *unique peak-period property* if the set $\{ t' : y^{t'}(pe) \ge y^t(pe)$ for all $t = 1, \ldots, T \}$ is a singleton.

In what follows we shall assume that p satisfies the unique peak-period property. Let τ be the unique time period such that $y^\tau(pe) > y^t(pe)$ for all other time periods (i.e., for all $t \in \{1, \ldots, T\} \setminus \{\tau\}$). Finally, we say that a full-capacity flat-rate equilibrium p satisfying unique peak-period property is *an exceptional case* if for all $t \in \{1, \ldots, T\} \setminus \{\tau\}$ it is the case that $p = \gamma_t = \gamma_\tau + \rho$.

3. Exceptional Case and Pareto Optimality

Consider a two-period economy with one consumer. We will denote the utility function of this lone consumer by u. Let $\omega > 0$ be the initial endowment of the numeraire good. Let $\gamma_2 = \gamma_1 + \rho$. The set of feasible activity vectors $Y = \{(y^1, y^2, -\lambda) \in \Re_+^2 \times (-\Re_+) : \lambda \ge \gamma_1 y^1 + \gamma_2 y^2 + \rho \max\{y^1, y^2\}\}$.

Let $u(x^1, x^2, \xi) = \alpha_1 \log x_1 + \alpha_2 \log x_2 + \alpha_0 \log \xi$, where $\alpha_t > 0$ for $t = 0, 1, 2$ and $\alpha_0 + \alpha_1 + \alpha_2 = 1$. Suppose $\alpha_1 > \alpha_2$.

At attainable allocations which are Pareto optimal it must be the case that (i) $y^t = x^t$ for $t = 1, 2$ and (ii) $\xi = \omega - \gamma_1 y^1 - \gamma_2 y^2 - \rho \max\{y^1, y^2\}$. Thus Pareto optimal allocations may be found by solving the following maximization problem:

$$\begin{aligned}&\text{maximize} && \alpha_1 \log x_1 + \alpha_2 \log x_2 + \alpha_0 \log \xi \\ &\text{subject to} && (\gamma_1+\rho)x_1 + (\gamma_1+\rho)x_2 + \xi \le \omega \\ &&& \gamma_1 x_1 + (\gamma_1 + 2\rho)x_2 + \xi \le \omega \\ &&& x_1, x_2, \xi \ge 0 .\end{aligned}$$

Note that for any $\xi \ge 0$, the two lines $(\gamma_1+\rho)x_1 + (\gamma_1+\rho)x_2 = \omega - \xi$ and $\gamma_1 x_1 + (\gamma_1 + 2\rho)x_2 = \omega - \xi$ intersect in the $x_1 x_2$ plane at the point ($\omega - \xi / 2(\tau_1 + \rho)$, $\omega - \xi / 2(\tau_1 + \rho)$). In this plane and above the 45^0 line through the origin, the peak period (of production) is 2, whereas below the 45^0 line through the origin the peak period is 1.

Given any $\xi \ge 0$, the unique solution to the problem:

$$\begin{aligned}&\text{maximize} && \alpha_1 \log x_1 + \alpha_2 \log x_2 \\ &\text{subject to} && (\gamma_1+\rho)x_1 + (\gamma_1+\rho)x_2 \le \omega - \xi \\ &&& \gamma_1 x_1 + (\gamma_1 + 2\rho)x_2 \le \omega - \xi \\ &&& x_1, x_2 \ge 0\end{aligned}$$

is at ($\alpha_1(\omega-\xi)/(\alpha_1+\alpha_2)(\gamma_1+\rho)$, $\alpha_2(\omega-\xi)/(\alpha_1+\alpha_2)(\gamma_1+\rho)$). The indifference curve through this point is tangent to the line $(\gamma_1+\rho)x_1 + (\gamma_1+\rho)x_2 \le \omega - \xi$ at the same point. Thus the consumption vector of the consumer at a Pareto optimal point is given by ($\alpha_1(\omega-\xi)/(\alpha_1+\alpha_2)(\gamma_1+\rho)$, $\alpha_2(\omega-\xi)/(\alpha_1+\alpha_2)(\gamma_1+\rho)$, ξ).

Hence the unique Pareto optimal point can be found by solving the following maximization problem:

$$\begin{aligned}&\text{maximize} && \alpha_1 \log x_1 + \alpha_2 \log x_2 + \alpha_0 \log \xi \\ &\text{subject to} && (\gamma_1+\rho)x_1 + (\gamma_1+\rho)x_2 + \xi \le \omega \\ &&& x_1, x_2, \xi \ge 0 .\end{aligned}$$

The unique solution to this problem is the point ($\alpha_1\omega/(\gamma_1+\rho)$, $\alpha_2\omega/(\gamma_1+\rho)$, $\alpha_0\omega$).

It is easily verified that $\gamma_1+\rho$ is a full-capacity flat-rate equilibrium with $(x((\gamma_1+\rho)e), \xi((\gamma_1+\rho)e)) = (\alpha_1\omega/(\gamma_1+\rho), \alpha_2\omega/(\gamma_1+\rho), \alpha_0\omega)$, $y((\gamma_1+\rho)e) = (\alpha_1\omega/(\gamma_1+\rho), \alpha_2\omega/(\gamma_1+\rho))$, and $\lambda = (1-\alpha_0)\omega$.

Since $\alpha_1 > \alpha_2$, 1 is the unique peak period. However $\gamma_1+\rho$ is obviously "an exceptional case."

It is easy to see that if a full-capacity flat-rate equilibrium satisfying the unique peak-period property is "an exceptional case" then the allocation associated with it must be Pareto optimal. This follows once we observe that for such a price the profit at no activity vector exceeds zero and the profit at any efficient activity vector is always zero. Thus the socially owned firm maximizes profit at the allocation associated with such an equilibrium price.

A price vector $\vec{p}$ is said to be *competitive* if for the allocation $[<(x(i, \vec{p}), \xi(i, \vec{p})) : i = 1, \ldots, m>, (y(\vec{p}), -\lambda(\vec{p}))]$, where $\lambda(\vec{p}) = \vec{\gamma} \cdot y(\vec{p}) +$

$\rho \max_{1 \le t \le T} y^t(\vec{p})$, it is the case that $\vec{p} \cdot y(\vec{p}) - \lambda(\vec{p}) \ge \vec{p} \cdot y - \lambda$ for all $(y, -\lambda) \in Y$.

It is easily observed that under such circumstances $\vec{p}$ is a full-capacity equilibrium (i.e., $\vec{p} \cdot y(\vec{p}) - \lambda(\vec{p}) = 0$). For if profits were positive at $(y(\vec{p}), \lambda(\vec{p}))$, then given constant returns to scale in production (as we have assumed), there would not exist any profit-maximizing activity vector. Thus from the note (in Section 2), it follows from the definition of a full-capacity equilibrium that the allocation is attainable (i.e., $\vec{\gamma} \cdot y(\vec{p}) + \rho \max_{1 \le t \le T} y^t(\vec{p}) + \sum_{i=1}^m \xi(i, \vec{p}) = \sum_{i=1}^m \omega(i)$). A *full-capacity flat-rate equilibrium* p is said to be *competitive* if the price vector $\vec{p} = pe$ is competitive.

If a full-capacity flat-rate equilibrium p satisfying the unique peak-period property is "an exceptional case," then $\vec{p}$ is competitive (i.e., $(pe - \vec{\gamma}) \cdot y(pe) - \rho \max_{1 \le t \le T} y^t(pe) \ge (pe) \cdot y - \lambda$ for all $(y, -\lambda) \in Y$).

The first fundamental theorem of welfare economics says that if a price vector $\vec{p}$ is competitive then the allocation $[< (x(i, \vec{p}), \xi(i, \vec{p})) : i = 1, \dots, m >, (y(\vec{p}), -\lambda(\vec{p}))]$ is Pareto optimal.

***Proposition 1*:** The allocation associated with a full-capacity flat-rate equilibrium satisfying unique peak-period property is Pareto optimal if the equilibrium is "an exceptional case."

References

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