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Calculating Value-at-Risk Using the Granularity Adjustment Method in the Portfolio Credit Risk Model with Random Loss Given Default

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According to the Basel Committee on Banking Supervision (BCBS), the internal ratings-based approach of Basel II and Basel III allows a bank to calculate the Value-at-Risk (VaR) for portfolio credit risk by using its own credit risk model. In this paper we use the Granularity Adjustment (GA) method proposed by Martin and Wilde (2002) to calculate VaR in the portfolio credit risk model with random loss given default. Moreover, we utilize a Monte Carlo simulation to study the impact of concentration risk on VaR.

Keywords: granularity adjustment method, loss given default, portfolio credit risk model, Value-at-Risk

JEL classification: C15, G32

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1 Introduction

In loan portfolios of a bank the main risk is the occurrence of defaults. A default in a loan portfolio means a borrower fails to meet its contractual obligation to repay a debt with the agreed terms. Loan portfolio defaults lead to huge losses for a bank, which is called portfolio credit risk. Under the Basel II and Basel III Accords, banks are allowed to establish their internal portfolio credit risk model to estimate their credit risk factor. The purpose of estimating this credit risk factor is to calculate regulatory capital for credit risk. This is called the internal ratings-based (IRB) approach of Basel II and Basel III. Under IRB of Basel II and Basel III, banks use the Value-at-Risk (VaR) to measure their portfolio credit risk and capital cushion. Thus, estimating VaR is an important issue.

Many articles have proposed several methods to calculate VaR. As a general rule, the Exposure at Default (EAD), Probability of Default (PD), and Loss Given Default (LGD) of each asset are input values to calculate VaR, where LGD is defined as the ratio between the actual loss and amount of loan in a default event. The earliest credit risk studies usually assume LGD to be fixed/constant (Vasicek, 1987, 1991, 2002; Emmer and Tasche, 2005) or random but independent of the default rate (Pykhtin and Dev, 2002; Gordy, 2003, 2004). However, many empirical studies point out that there is a strong correlation between LGD and the default rate. Thus, LGD should be random and correlated with the default rate (Frye, 2000; Andersen and Sidenius 2004; Altman *et al.* 2005; Bruche and González-Aguado, 2010; Van Damme, 2011; Farinelli and Shkolnikov, 2012). For simplicity, the uncorrelated random LGD model represents that LGD is random and uncorrelated with the default rate, and the correlated random LGD model represents that LGD is random and correlated with the default rate.

Vasicek (1987, 1991, 2002) use the law of large numbers method to calculate VaR under the asymptotic single risk factor (ASRF) assumption with constant LGD. The ASRF approach assumes that the portfolio is infinitely fine grained and only one systematic risk factor could affect the default risk of all assets in the portfolio. However, the ASRF framework cannot capture the concentration risk that affects the accuracy of the estimate for VaR. For more details about the ASRF approach, please refer to Gordy (2003, 2004) and Overbeck and Wanger (2003). Wilde (2001) and Martin and Wilde (2002) propose a Granularity Adjustment (GA) method to calculate VaR by using the Taylor expansion of the quantile and the results of Gouriéroux *et al.* (2000). The impact of concentration risk on VaR can be approximated analytically through the GA method.

Pykhtin and Dev (2002), Emmer and Tasche (2005), and Bellalah *et al.* (2015) use the GA method to calculate VaR in the portfolio credit risk model with random and uncorrelated LGD and constant LGD, respectively. Gordy (2003, 2004) also employs the GA method to calculate VaR in the CreditRisk$^+$ model with random and uncorrelated LGD. Gürtler *et al.* (2010) make an extension of the GA method to obtain the closed form of VaR in the multi-factor model with constant LGD. Note that these studies do not consider the random and correlated LGD for calculating VaR. Lin (2010) utilize the GA method to calculate VaR in the portfolio credit risk model with random and correlated LGD.

The main contribution of this paper is that we show that the GA method can successfully calculate VaR in portfolio credit risk with stochastic and correlated LGD. We also use a Monte Carlo simulation to study how the concentration risk affects VaR. Note that, in general, there are two types of concentration risk. One is referred to as single-name concentration, i.e., the portfolio with a large EAD on highly rated obligors. The other one is referred to as sectoral concentration, i.e., EAD to obligors of the portfolio in the same sectors. For more studies on the concentration risk of portfolio, please refer to Lütkebohmert (2009). In this study we focus on how the single-name concentration affects VaR. Emmer and Tasche (2005) also make a similar study with constant LGD.

This paper is organized as follows. Section 2 introduces the portfolio credit risk model and description of VaR. Section 3 describes the approximate closed form of VaR with random and correlated LGD. Section 4 shows the Monte Carlo simulation results. Section 5 summarizes the conclusions. Finally, Appendix presents all proofs of the lemmas.

2 Portfolio Credit Risk Model and VaR

Consider a portfolio with m assets. We define the standardized asset value X_i of the ith asset in the portfolio as:

$$X_i = \sqrt{\rho_i} Z + \sqrt{1-\rho_i} U_i, \quad i = 1, \cdots, m, \tag{1}$$

where $0 \leq \rho_i \leq 1$, and $Z, U_1, \cdots, U_m$ be i.i.d. $\mathcal{N}(0,1)$. By a simple computation, X_i also follow $N(0,1)$, and $\sqrt{\rho_i \rho_j}$ denotes the correlation between X_i and X_j $(i \neq j)$. Let Y_i represent the default indicator function of the ith asset:

$$Y_i = I(X_i \leq c_i), \tag{2}$$

where c_i is the default threshold and $I(\cdot)$ is an indicator function. Asset i is assumed to be in default when its standardized asset value X_i falls below the threshold value c_i. We define the probability of default of asset i as PD_i:

$$\begin{aligned} PD_i &\equiv \mathsf{P}(Y_i = 1) \\ &= \mathsf{P}(X_i < c_i) \\ &= \Phi(c_i), \end{aligned}$$

where $\Phi(\cdot)$ is the cumulative distribution function of $\mathcal{N}(0,1)$, and thus:

$$c_i = \Phi^{-1}(PD_i),$$

where $\Phi^{-1}(\cdot)$ is the inverse function of $\Phi(\cdot)$.

In this paper LGD is random and correlated, as proposed by Andersen and Sidenius (2004), i.e., LGD_i of the ith asset is:

$$LGD_i = 1 - \Phi(u_i + \sigma_i \eta_i), \tag{3}$$

where $-\infty < u_i < \infty$, $\sigma_i > 0$. The random variables η_i are assumed to be:

$$\eta_i = \sqrt{\lambda_i} Z + \sqrt{1-\lambda_i}\, \varepsilon_i, \tag{4}$$

where $0 \le \lambda_i \le 1$, and $Z, \varepsilon_1, \cdots, \varepsilon_m$ be i.i.d. $\mathcal{N}(0,1)$. Note that η_i also follows $\mathcal{N}(0,1)$. When $\lambda_i = 0$, $i = 1, \cdots, m$, LGD is random and uncorrelated with the default rate and called the uncorrelated stochastic LGD model. When $\lambda_i > 0$, $i = 1, \cdots, m$, LGD is random and correlated with the default rate and called the correlated random LGD model.

Lemma 1. Under the correlated random LGD model, we have:

$$\begin{aligned}
\mathsf{E}(LGD_i) &= \Phi\left(\frac{-u_i}{\sqrt{1+\sigma_i^2}}\right), \\
\mathsf{Var}(LGD_i) &= \Phi_2\left(\frac{-u_i}{\sqrt{1+\sigma_i^2}}, \frac{-u_i}{\sqrt{1+\sigma_i^2}}; \frac{\sigma_i^2}{1+\sigma_i^2}\right) \\
&\quad -\Phi\left(\frac{-u_i}{\sqrt{1+\sigma_i^2}}\right)^2, \\
\mathsf{Cov}(LGD_i, LGD_j) &= \Phi_2\left(\frac{-u_i}{\sqrt{1+\sigma_i^2}}, \frac{-u_j}{\sqrt{1+\sigma_j^2}}; \frac{\sigma_i \sigma_j \sqrt{\lambda_i \lambda_j}}{\sqrt{(1+\sigma_i^2)(1+\sigma_j^2)}}\right) \\
&\quad -\Phi\left(\frac{-u_i}{\sqrt{1+\sigma_i^2}}\right)\Phi\left(\frac{-u_j}{\sqrt{1+\sigma_j^2}}\right), \\
\mathsf{Cov}(LGD_i, Y_i) &= \Phi_2\left(\frac{-u_i}{\sqrt{1+\sigma_i^2}}, \Phi^{-1}(PD_i); \frac{\sigma_i \sqrt{\lambda_i \rho}}{\sqrt{1+\sigma_i^2}}\right)
\end{aligned}$$

$$
\begin{aligned}
& -\Phi\left(\frac{-u_i}{\sqrt{1+\sigma_i^2}}\right)\times PD_i, \\
\mathsf{Corr}(LGD_i, LGD_j) &= \frac{\mathsf{Cov}(LGD_i, LGD_j)}{\sqrt{\mathsf{Var}(LGD_i)\mathsf{Var}(LGD_j)}}, \\
\mathsf{Corr}(LGD_i, Y_i) &= \frac{\mathsf{Cov}(LGD_i, Y_i)}{\sqrt{\mathsf{Var}(LGD_i)\mathsf{Var}(Y_i)}},
\end{aligned}
$$

where $\mathsf{Var}(Y_i) = PD_i - PD_i^2$.

Consider a portfolio with m assets. We denote EAD and LGD of the ith asset as EAD_i and LGD_i, respectively. We discuss the VaR specifications as follows. VaR is the value in which a loss on a portfolio will not exceed a given value over the given time horizon with a level of probability. More precisely, we define the loss rate of a portfolio as:

$$L_m = \sum_{i=1}^{m} w_i \times LGD_i \times Y_i, \tag{5}$$

where

$$w_i = \frac{EAD_i}{\sum_{i=1}^{m} EAD_i}$$

denotes the weight of EAD of the ith asset. When L_m is a random variable and the confidence level is: α $(0 < \alpha < 1)$, the α quantile of L_m is:

$$q_\alpha(L_m) = \inf\{\ell \geq 0 : \mathsf{P}(L_m \leq \ell) \geq \alpha\}.$$

Under the IRB approach of Basel II and Basel III, α is taken as 99.9%. We then denote the α-quantile of L_m as VaR:

$$\text{VaR} = q_{99.9\%}(L_m).$$

3 Calculating Portfolio Credit VaR Using the GA Method

Martin and Wilde (2002) use the Taylor expansion and results from Gouriéroux *et al.* (2000) to obtain the portfolio credit VaR with the GA method in more general model specifications. For convenience, we let:

$$g(z) = \mathsf{E}(L_m|Z = z)$$

be the conditional expectation of L_m given $Z = z$.

Condition 1. Given the realization of $Z = z$, $g(z)$ is a continuous, differentiable, and decreasing function.

Condition 2. Given the realization of $Z = z$, $\mathsf{Var}(L_m|Z = z)$ is a continuous and differentiable function.

Bluhm *et al.* (2003) employ the law of large numbers method to obtain the following results under the ASRF assumption, i.e.:

$$\begin{aligned} \alpha &= \mathsf{P}(L_m \leq q_\alpha(L_m)) \\ &\approx \mathsf{P}(\mathsf{E}(L_m|Z) \leq q_\alpha(L_m)). \end{aligned}$$

If condition 1 holds, then:

$$\begin{aligned} q_\alpha(L_m) &\approx \mathsf{E}(L_m|Z = z_\alpha) \\ &= g(z_\alpha), \end{aligned}$$

where z_α is the $(1-\alpha)$ quantile of the standard normal distribution. For simplicity, using $g(z_\alpha)$ to estimate VaR is called the ASRF method. According to the results of Martin and Wilde (2002), under the correlated random LGD model, the approximate α quantile of L_m is:

$$q_\alpha(L_m) \approx g(z_\alpha) + GA, \tag{6}$$

where

$$GA = -\frac{1}{2g'(z)} \left\{ \frac{\partial}{\partial z} \mathsf{Var}(L_m|Z = z) \right.$$

$$-\mathsf{Var}(L_m|Z=z)\left[\frac{g''(z)}{g'(z)}+z\right]\Bigg\}\Bigg|_{z=z_\alpha}. \tag{7}$$

Lemma 2. Under the correlated random LGD model, $g(z)$ is given by:

$$g(z)=\sum_{i=1}^{m} w_i \Phi(\psi_i(z))\Phi(\varsigma_i(z)),$$

where

$$\psi_i(z) = \frac{-u_i-\sigma_i\sqrt{\lambda_i}z}{\sqrt{1+\sigma_i^2(1-\lambda_i)}}, \tag{8}$$

$$\varsigma_i(z) = \frac{\Phi^{-1}(PD_i)-\sqrt{\rho_i}\,z}{\sqrt{1-\rho_i}}. \tag{9}$$

Through straightforward calculus, we can obtain $g'(z)$ and $g''(z)$ as the following.

Lemma 3. Under the correlated random LGD model, $g'(z)$ and $g''(z)$ are given by:

$$\begin{aligned}
g'(z) &= \sum_{i=1}^{m} w_i\tilde{\psi}_i\phi(\psi_i(z))\Phi(\varsigma_i(z))+\sum_{i=1}^{m} w_i\tilde{\varsigma}_i\Phi(\psi_i(z))\phi(\varsigma_i(z)),\\
g''(z) &= -\sum_{i=1}^{m} w_i\tilde{\psi}_i^2\psi_i(z)\phi(\psi_i(z))\Phi(\varsigma_i(z))+2\sum_{i=1}^{m} w_i\tilde{\psi}_i\tilde{\varsigma}_i\phi(\psi_i(z))\phi(\varsigma_i(z))\\
&\quad -\sum_{i=1}^{m} w_i\tilde{\varsigma}_i^2\varsigma_i(z)\Phi(\psi_i(z))\phi(\varsigma_i(z)),
\end{aligned}$$

where $\phi(\cdot)$ is the probability density function of the standard normal distribution:

$$\begin{aligned}
\tilde{\psi}_i &= \frac{\partial}{\partial z}\psi_i(z)=\frac{-\sigma_i\sqrt{\lambda_i}}{\sqrt{1+\sigma_i^2(1-\lambda_i)}},\\
\tilde{\varsigma}_i &= \frac{\partial}{\partial z}\varsigma_i(z)=-\sqrt{\frac{\rho_i}{1-\rho_i}}.
\end{aligned}$$

According to Lemma 3, we observe that $g'(z)<0$. Thus, condition 1 holds.

Lemma 4. Under the correlated random LGD model, $\mathsf{Var}(L_m|Z=z)$ and $\frac{\partial}{\partial z}\mathsf{Var}(L_m|Z=z)$ are given by:

$$\begin{aligned}\mathsf{Var}(L_m|Z=z) &= \sum_{i=1}^{m} w_i^2 \Phi_2(\psi_i(z),\psi_i(z);\rho_i^*)\Phi(\varsigma_i(z)) \\ &\quad - \sum_{i=1}^{m} w_i^2 \Phi^2(\psi_i(z))\Phi^2(\varsigma_i(z)),\end{aligned}$$

and

$$\begin{aligned}&\frac{\partial}{\partial z}\mathsf{Var}(L_m|Z=z) \\ &= 2\sum_{i=1}^{m} w_i^2 \tilde{\psi}_i \phi(\psi_i(z))\,\Phi\left(\frac{\psi_i(z)}{\sqrt{1+2\sigma_i^2(1-\lambda_i)}}\right)\Phi(\varsigma_i(z)) \\ &\quad + \sum_{i=1}^{m} w_i^2 \tilde{\varsigma}_i \Phi_2(\psi_i(z),\psi_i(z);\rho_i^*)\phi(\varsigma_i(z)) \\ &\quad -2\sum_{i=1}^{m} w_i^2 [\tilde{\psi}_i\phi(\psi_i(z))\Phi(\varsigma_i(z)) + \tilde{\varsigma}_i\Phi(\psi_i(z))\phi(\varsigma_i(z))] \\ &\quad \times\Phi(\psi_i(z))\Phi(\varsigma_i(z)),\end{aligned}$$

where $\Phi_2(\cdot,\cdot;\rho_i^*)$ is the standard bivariate normal cumulative distribution with correlation

$$\rho_i^* = \frac{\sigma_i^2(1-\lambda_i)}{1+\sigma_i^2(1-\lambda_i)}. \tag{10}$$

Lastly, we obtain the approximate closed form of $q_\alpha(L_m)$ by using (6), (7), and Lemmas 3 and 4. For simplicity, using the approximate closed form of $q_\alpha(L_m)$ to estimate VaR is called the GA method.

4 Simulation Results

The Monte Carlo studies present how the concentration risk of a portfolio affects VaR. For more studies on the concentration risk of a portfolio, please refer to Lütkebohmert

(2009). According to the rule of Basel II and Basel III (BCBS, 2011), the value of ρ_i is set to be:

$$\rho_i = 0.12 \left[\frac{1-\exp(-50 \times PD_i)}{1-\exp(-50)} \right] + 0.24 \left[1 - \frac{1-\exp(-50 \times PD_i)}{1-\exp(-50)} \right].$$

This paper takes the Monte Carlo simulation (MCS) method as a benchmark method to calculate VaR. We give the steps of the MCS method as follows.

Step 1: Simulate $m+1$ random samples $Z, U_1, \cdots, U_m$ from $\mathcal{N}(0,1)$ and obtain $X_1, \cdots, X_m$ from (1).

Step 2: Take $X_1, \cdots, X_m$ into (2) to obtain the default indicator functions $Y_1, \cdots, Y_m$.

Step 3: Simulate random samples $\varepsilon_1, \cdots, \varepsilon_m$ from $\mathcal{N}(0,1)$ and obtain $\eta_1, \cdots, \eta_m$ from (4). One can then obtain $LGD_1, \cdots, LGD_m$ from (3).

Step 4: Obtain $L_1, \cdots, L_m$ from (5).

Step 5: Repeat Steps 1-4 10^6 times.

Step 6: Take the $10^6 \times \alpha$-largest of simulated L_m as the estimate for $q_\alpha(L_m)$.

In the simulation studies, we focus on how the single-name concentration affects VaR. Emmer and Tasche (2005) also conduct a similar study with constant LGD. Thus, the choice of parameter settings in this paper is the same as that in the simulation of Emmer and Tasche (2005). The set-up for number of assets, PD, and EAD are given as follows.

1. $m = 1000$, $\alpha = 0.999$.
2. $PD_1 = 0.002$, $PD_2 = \cdots = PD_m = 0.025$, i.e., the first asset has a smaller PD than other assets.
3. Given the weight of EAD of first asset w_1, assume:

$$w_2 = \cdots = w_m = \frac{1-w_1}{m-1},$$

i.e., the weight of EAD of assets is the same except for the first asset. The range of w_1 is $[0, 0.2]$. When $w_1 = 1/m$, $w_2 = \cdots = w_m = 1/m$, it means that there is no concentration risk in portfolio. When $w_1 > 1/m$ and w_1 becomes large, the concentration risk becomes large.

4. $\mathsf{E}(LGD_i) = 0.6$, $i = 1, \cdots, m$.
5. $\mathsf{SD}(LGD_i) = \sqrt{\mathsf{Var}(LGD_i)} = 0, 0.2, 0.4$, $i = 1, \cdots, m$.
6. $\mathsf{Corr}(LGD_i, LGD_j) = 0, 0.3, 0.6$, $i \neq j$, $i, j = 1, \cdots, m$. When $\mathsf{Corr}(LGD_i, LGD_j) = 0$, $i \neq j$, LGD is random and uncorrelated. When $\mathsf{Corr}(LGD_i, LGD_j) > 0$, $i \neq j$, LGD is random and correlated.

Note that a constant LGD is assumed in the simulation studies of Emmer and Tasche (2005), namely - $LGD_i = 1$, $i = 1, \cdots, m$.

Figure 1 shows VaRs estimated by MCS, GA, and ASRF methods in the constant LGD model. Figure 2 illustrates VaRs estimated by MCS, GA, and ASRF methods in the correlated random LGD model with $\mathsf{SD}(LGD_i) = 0.2$ and $\mathsf{SD}(LGD_i) = 0.4$. Note that when $\mathsf{SD}(LGD_i) = 0$, LGD is constant. When $\mathsf{SD}(LGD_i) > 0$, LGD is random.

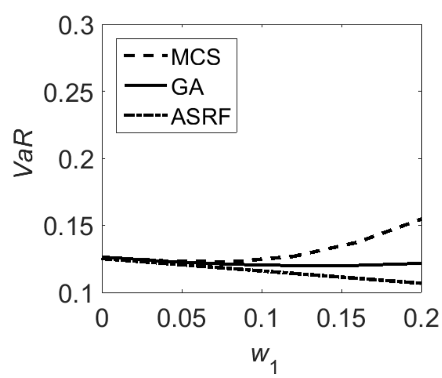

Figure 1: Estimated Portfolio VaRs by Using MCS, GA, and ASRF Methods under the Constant LGD Model

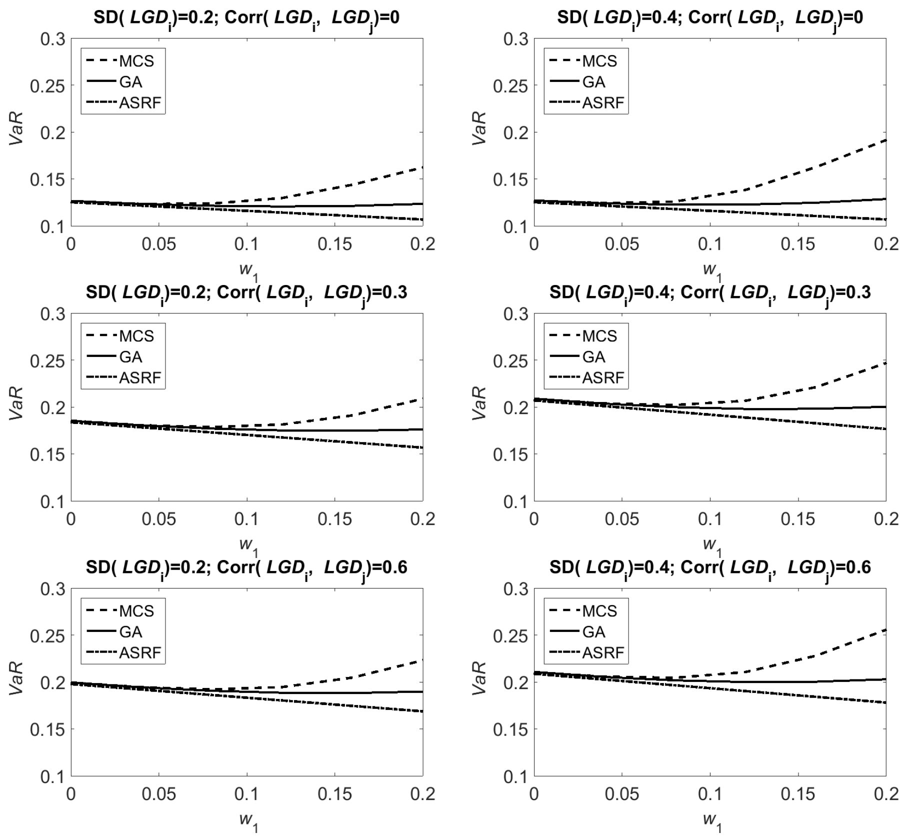

Figure 2: Estimated Portfolio VaRs by Using MCS, GA, and ASRF Methods under the Correlated Random LGD Model

Several conclusions can be observed in Figures 1-2.

1. When the weight of EAD of first asset w_1 is increasing from $1/m$ to 1, the trend of VaR will first be smaller and then larger in both the constant and correlated random LGD models. In fact, the same conclusion is also obtained in Emmer and Tasche (2005).

2. The performance of the estimated VaR using the GA method is better than the ASRF method in both the constant and correlated random LGD models. However, as $w > 0.1$, the estimated VaRs have a larger error by using the GA and the ASRF methods. As w_1 becomes large, the error increases. In other words, when the weight of EAD of the first asset is larger than 10%, the estimated VaR using the GA method has a larger error and significantly underestimates the risk. In fact, the same conclusion is also obtained in Emmer and Tasche (2005).
3. The estimated portfolio VaR under the correlated random LGD model is larger than the estimated VaR under the constant LGD model. This means that, as the true LGD is random, the estimated VaR by using the constant LGD model underestimates the risk.
4. The simulation results also show the economic/management meaning of portfolio allocations. Fund managers and investors can thus carefully allocate their portfolio to decrease the concentration of assets. Due to the concentration of assets, the portfolio will incur a high concentration risk and VaR.

5 Conclusions

The major work of this paper is to obtain the approximate closed form of VaR by using the GA method proposed by Martin and Wilde (2002) under the correlated random LGD model. The results improve the analysis presented in Emmer and Tasche (2005). We observe that the VaR performance using the GA method is better than the performance using the ASRF method from our simulation results. However, we note that when the weight of EAD of one asset is large, the estimated VaR using the GA method has a larger error and significantly underestimates the risk.

Appendix

This section contains all proofs of lemmas. We first present the three lemmas herein.

Lemma A1 (Andersen and Sidenius (2004)). Given the constants a and b:

$$\int_{-\infty}^{\infty} \Phi(ax+b)\phi(x)\,dx = \Phi\left(\frac{b}{\sqrt{1+a^2}}\right).$$

Lemma A2 (Andersen and Sidenius (2004)). Given the constants a_1, a_2, b_1, and b_2:

$$\begin{aligned}
&\int_{-\infty}^{\infty} \Phi(a_1x+b_1)\Phi(a_2x+b_2)\phi(x)\,dx \\
&= \Phi_2\left(\frac{b_1}{\sqrt{1+a_1^2}}, \frac{b_2}{\sqrt{1+a_2^2}}; \frac{a_1a_2}{\sqrt{(1+a_1^2)(1+a_2^2)}}\right).
\end{aligned}$$

Lemma A3. Given the constants a_1, a_2, b_1, and b_2:

$$\begin{aligned}
&\int_{-\infty}^{\infty} \Phi(a_1x+b_1)\phi(a_2x+b_2)\phi(x)\,dx \\
&= \frac{1}{\sqrt{1+a_2^2}}\phi\left(\frac{b_2}{\sqrt{1+a_2^2}}\right)\Phi\left(\frac{b_1+a_2^2b_1-a_1a_2b_2}{\sqrt{(1+a_2^2)(1+a_1^2+a_2^2)}}\right).
\end{aligned}$$

Proof. By Lemma A1:

$$\begin{aligned}
&\int_{-\infty}^{\infty} \Phi(a_1x+b_1)\phi(a_2x+b_2)\phi(x)dx \\
&= \phi\left(\frac{b_2}{\sqrt{1+a_2^2}}\right)\int_{-\infty}^{\infty} \Phi(a_1x+b_1)\phi\left(\sqrt{1+a_2^2}x+\frac{a_2b_2}{\sqrt{1+a_2^2}}\right)dx \\
&= \frac{1}{\sqrt{1+a_2^2}}\phi\left(\frac{b_2}{\sqrt{1+a_2^2}}\right)
\end{aligned}$$

$$\times \int_{-\infty}^{\infty} \Phi\left(\frac{a_1}{\sqrt{1+a_2^2}} y + \frac{b_1 + a_2^2 b_1 - a_1 a_2 b_2}{1+a_2^2}\right) \phi(y)\, dy$$

$$= \frac{1}{\sqrt{1+a_2^2}} \phi\left(\frac{b_2}{\sqrt{1+a_2^2}}\right) \Phi\left(\frac{b_1 + a_2^2 b_1 - a_1 a_2 b_2}{\sqrt{(1+a_2^2)(1+a_1^2+a_2^2)}}\right).$$

Proof of Lemma 1.

By Lemma A1:

$$\begin{aligned} \mathsf{E}(LGD_i) &= \int_{-\infty}^{\infty} [1 - \Phi(u_i + \sigma_i \eta_i)] \phi(\eta_i)\, d\eta_i \\ &= \int_{-\infty}^{\infty} \Phi(-u_i - \sigma_i \eta_i) \phi(\eta_i)\, d\eta_i \\ &= \Phi\left(\frac{-u_i}{\sqrt{1+\sigma_i^2}}\right). \end{aligned} \tag{11}$$

By Lemma A1, Lemma A2, and (11):

$$\begin{aligned} \mathsf{Var}(LGD_i) &= \mathsf{E}\left[\Phi^2(-u_i - \sigma_i \eta_i)\right] - \mathsf{E}^2(LGD_i) \\ &= \int_{-\infty}^{\infty} \Phi^2(-u_i - \sigma_i \eta_i) \phi(\eta_i)\, d\eta_i - \mathsf{E}^2(LGD_i) \\ &= \Phi_2\left(\frac{-u_i}{\sqrt{1+\sigma_i^2}}, \frac{-u_i}{\sqrt{1+\sigma_i^2}}; \frac{\sigma_i^2}{1+\sigma_i^2}\right) - \Phi^2\left(\frac{-u_i}{\sqrt{1+\sigma_i^2}}\right). \end{aligned}$$

By Lemma A2 and (11):

$$\begin{aligned} &\mathsf{Cov}(LGD_i, LGD_j) \\ &= \mathsf{E}[\mathsf{E}(LGD_i \times LGD_j | Z)] - \mathsf{E}(LGD_i) \times \mathsf{E}(LGD_j) \\ &= \mathsf{E}\Bigg[\int_{-\infty}^{\infty} \Phi\Big(-u_i - \sigma_i \left(\sqrt{\lambda_i} Z + \sqrt{1-\lambda_i}\, \varepsilon_i\right)\Big) \phi(\varepsilon_i)\, d\varepsilon_i \\ &\quad \times \int_{-\infty}^{\infty} \Phi\Big(-u_j - \sigma_j \left(\sqrt{\lambda_j} Z + \sqrt{1-\lambda_j}\, \varepsilon_j\right)\Big) \phi(\varepsilon_j)\, d\varepsilon_j\Bigg] \end{aligned}$$

$$
\begin{aligned}
& -\mathsf{E}(LGD_i)\times\mathsf{E}(LGD_j) \\
= \;& \int_{-\infty}^{\infty}\Phi\left(\frac{-u_i-\sigma_i\sqrt{\lambda_i}\,z}{\sqrt{1+\sigma_i^2(1-\lambda_i)}}\right)\Phi\left(\frac{-u_j-\sigma_j\sqrt{\lambda_j}\,z}{\sqrt{1+\sigma_j^2(1-\lambda_j)}}\right)\phi(z)\,dz \\
& -\mathsf{E}(LGD_i)\times\mathsf{E}(LGD_j) \\
= \;& \Phi_2\left(\frac{-u_i}{\sqrt{1+\sigma_i^2}},\frac{-u_j}{\sqrt{1+\sigma_j^2}};\frac{\sigma_i\sigma_j\sqrt{\lambda_i\lambda_j}}{\sqrt{(1+\sigma_i^2)(1+\sigma_j^2)}}\right) \\
& -\Phi\left(\frac{-u_i}{\sqrt{1+\sigma_i^2}}\right)\Phi\left(\frac{-u_j}{\sqrt{1+\sigma_j^2}}\right).
\end{aligned}
$$

By Lemma A2 and (11):

$$
\begin{aligned}
& \mathsf{Cov}(LGD_i,Y_i) \\
= \;& \mathsf{E}[\mathsf{E}(LGD_i\times Y_i|Z)]-\mathsf{E}(LGD_i)\times\mathsf{E}(Y_i) \\
= \;& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi\Big(-u_i-\sigma_i\left(\sqrt{\lambda_i}Z+\sqrt{1-\lambda_i}\,\varepsilon_i\right)\Big)\phi(\varepsilon_i) \\
& \qquad\qquad \times\Phi\left(\frac{\Phi^{-1}(PD_i)-\sqrt{\rho_i}Z}{\sqrt{1-\rho_i}}\right)d\varepsilon_i\,dz-\mathsf{E}(LGD_i)\times\mathsf{E}(Y_i) \\
= \;& \int_{-\infty}^{\infty}\Phi\left(\frac{-u_i-\sigma_i\sqrt{\lambda_i}\,z}{\sqrt{1+\sigma_i^2(1-\lambda_i)}}\right)\Phi\left(\frac{\Phi^{-1}(PD_i)-\sqrt{\rho_i}\,Z}{\sqrt{1-\rho_i}}\right)\phi(z)\,dz \\
& -\mathsf{E}(LGD_i)\times\mathsf{E}(Y_i) \\
= \;& \Phi_2\left(\frac{-u_i}{\sqrt{1+\sigma_i^2}},\Phi^{-1}(PD_i);\frac{\sigma_i\sqrt{\lambda_i\rho}}{\sqrt{1+\sigma_i^2}}\right)-\Phi\left(\frac{-u_i}{\sqrt{1+\sigma_i^2}}\right)\times PD_i.
\end{aligned}
$$

Proof of Lemma 2.

Given $Z=z$, the conditional default probability of the ith asset is:

$$
\mathsf{P}(Y_i=1|Z=z) \;=\; \mathsf{P}(X_i<\Phi^{-1}(PD_i)|Z=z)
$$

$$
\begin{aligned}
&= \mathsf{P}\left(U_i < \frac{\Phi^{-1}(PD_i) - \sqrt{\rho_i} Z}{\sqrt{1-\rho_i}} \middle| Z = z \right) \\
&= \Phi(\varsigma_i(z)),
\end{aligned}
$$

where $\varsigma_i(z)$ is defined in (9). By Lemma 3:

$$
\begin{aligned}
g(z) &= \mathsf{E}(L_m | Z = z) \\
&= \mathsf{E}\left(\sum_{i=1}^{m} w_i \times LGD_i \times Y_i \middle| Z = z \right) \\
&= \sum_{i=1}^{m} w_i \, \mathsf{E}[\Phi(-u_i - \sigma_i \eta_i) | Z = z] \, \mathsf{E}(Y_i | Z = z) \\
&= \sum_{i=1}^{m} w_i \left[\int_{-\infty}^{\infty} \Phi\Big(-u_i - \sigma_i \left(\sqrt{\lambda_i} z + \sqrt{1-\lambda_i}\, \varepsilon_i \right) \Big) \phi(\varepsilon_i) \, d\varepsilon_i \right] \\
&\quad \times \mathsf{P}(Y_i = 1 | Z = z) \\
&= \sum_{i=1}^{m} w_i \Phi(\psi_i(z)) \Phi(\varsigma_i(z)),
\end{aligned}
$$

where $\psi_i(z)$ and $\varsigma_i(z)$ are defined in (8) and (9), respectively.

Proof of Lemma 4.

By Lemma A2:

$$
\begin{aligned}
&\mathsf{Var}(L_m | Z = z) \\
&= \mathsf{Var}\left(\sum_{i=1}^{m} w_i \times LGD_i \times Y_i \middle| Z = z \right) \\
&= \sum_{i=1}^{m} w_i^2 \, \mathsf{E}(LGD_i^2 \times Y_i^2 | Z = z) - \sum_{i=1}^{m} w_i^2 \, \mathsf{E}^2(LGD_i \times Y_i | Z = z) \\
&= \sum_{i=1}^{m} w_i^2 \left[\int_{-\infty}^{\infty} \Phi^2\Big(-u_i - \sigma_i \left(\sqrt{\lambda_i} z + \sqrt{1-\lambda_i}\, \varepsilon_i \right) \Big) \phi(\varepsilon_i) d\varepsilon_i \right] \\
&\quad \times \Phi(\varsigma_i(z)) \\
&\quad - \sum_{i=1}^{m} w_i^2 \left[\int_{-\infty}^{\infty} \Phi\Big(-u_i - \sigma_i \left(\sqrt{\lambda_i} z + \sqrt{1-\lambda_i}\, \varepsilon_i \right) \Big) \phi(\varepsilon_i) d\varepsilon_i \right]^2
\end{aligned}
$$

$$\times\Phi^2(\varsigma_i(z))$$
$$= \sum_{i=1}^{m} w_i^2 \Phi_2(\psi_i(z), \psi_i(z); \rho^*)\Phi(\varsigma_i(z)) - \sum_{i=1}^{m} w_i^2 \Phi^2(\psi_i(z))\Phi^2(\varsigma_i(z)),$$

where $\psi_i(z)$, $\varsigma_i(z)$, and ρ^* are defined in (8), (9), and (10), respectively. Moreover:

$$\begin{aligned}
&\frac{\partial}{\partial z}\Phi_2(\psi_i(z), \psi_i(z); \rho^*) \\
&= \frac{\partial}{\partial z}\int_{-\infty}^{\infty} \Phi^2\Big(-u_i - \sigma_i\left(\sqrt{\lambda_i}\, z + \sqrt{1-\lambda_i}\,\varepsilon_i\right)\Big)\,\phi(\varepsilon_i)\, d\varepsilon_i \\
&= -2\sigma_i\sqrt{\lambda_i}\int_{-\infty}^{\infty} \Phi\Big(-u_i - \sigma_i\sqrt{\lambda_i}\, z - \sigma_i\sqrt{1-\lambda_i}\,\varepsilon_i\Big) \\
&\qquad\qquad \times\phi\Big(-u_i - \sigma_i\sqrt{\lambda_i}\, z - \sigma_i\sqrt{1-\lambda_i}\,\varepsilon_i\Big)\,\phi(\varepsilon_i)\, d\varepsilon_i \\
&= 2\tilde{\psi}_i\phi(\psi_i(z))\Phi\left(\frac{\psi_i(z)}{\sqrt{1+2\sigma_i^2(1-\lambda_i)}}\right).
\end{aligned}$$

By a straightforward computation, Lemma 4 can be proved.

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