Conditional Diagnosability of the BC Networks under the Comparison Diagnosis Model*+

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Abstract- *An n-dimensional bijective connection network (BC network), denoted by X_n, is an n-regular graph with 2^n vertices and $n2^{n-1}$ edges. The n-dimensional hypercube, crossed cube, twisted cube, and Möbius cube are some examples of the n-dimensional BC networks. In [5], Lai et al. introduced a novel measure of diagnosability, called conditional diagnosability, by adding an additional condition that any faulty set cannot contain all the neighbors of any vertex in a system. In this paper, we prove that the conditional diagnosability of X_n is 3(n-2)+1 under the comparison model, $n \geq 5$. As a corollary of this result, we obtain the conditional diagnosability of the hypercubes, crossed cubes, twisted cubes, and Möbius cubes.*

Keywords: comparison diagnosis model, diagnosability, conditional diagnosability, BC network.

1. Introduction

The problem of fault diagnosis in multiprocessor systems has gained increasing importance and has been widely studied in the literatures [2], [3], [5], [6], [11], [13]. In order to diagnose a multiprocessor system, several different models have been proposed [7], [9]. Throughout this paper, we base our diagnosability analysis on the comparison model. The comparison model deals with the faulty diagnosis by sending the same input (or task) from a vertex w to each pair of distinct neighbors, u and v, and then comparing their responses. The vertex w is called the *comparator* of vertices u and v. The result of the comparison is either the two responses agreed or two responses disagreed. Based on the results of all the comparisons, the system can decide the faulty or fault-free status of the vertices.

Reviewing some previous papers [2], [3], [6], [11], the Hypercube Q_n, the Crossed cube CQ_n, the Twisted cube TQ_n, and the Möbius cube MQ_n, all have diagnosability n under the comparison model. In classical measures of system-level diagnosability for multiprocessor systems, if all the neighbors of some processor v are faulty simultaneously, it is not possible to determine whether processor v is fault-free or faulty. As a consequence, the diagnosability of a system is limited by its minimum degree. Therefore, Lai et al. introduced a restricted diagnosability of multiprocessor systems called *conditional diagnosability* in [5]. Lai et al. considered a measure by restricting that, for each processor v in a system, not all the processors which are directly connected to v fail at the same time. In this paper, we prove that the conditional diagnosability of n-dimensional BC networks X_n is 3(n-2)+1 under the comparison model, $n \geq 5$. As a corollary of this result, we obtain the conditional diagnosability of the hypercubes, crossed cubes, twisted cubes, and Möbius cubes.

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2. Preliminaries

For the graph definition and notation we follow [12]. A multiprocessor system can be modeled as a graph $G(V,E)$, where the set of vertices V represents processors and the set of edges E represents communication links between processors.

Let $G(V,E)$ be a graph and $v \in V(G)$ be a vertex. The neighborhood $N(v)$ of vertex v is the set of all vertices that are adjacent to v. The cardinality $|N(v)|$ is called the degree of v, denoted by $deg_G(v)$ or simply $deg(v)$. For a subset of vertices $V' \subset V(G)$, the neighborhood set of the vertex set V' is defined as $N(V')=\bigcup_{v \in V'} N(v) - V'$. For a set of vertices(respectively, edges) S, we use the notation $G - S$ to denote the graph obtained from G by removing all the vertices(respectively, edges) in S. The components of a graph G are its maximal connected subgraphs. A component is trivial if it has no edges; otherwise, it is nontrivial. The connectivity $\kappa(G)$ of a graph $G(V,E)$ is the minimum number of vertices whose removal results in a disconnected or a trivial graph. Let $F_1,F_2 \subseteq V(G)$ be two distinct sets. The symmetric difference of the two sets F_1 and F_2 is defined as the set $F_1 \Delta F_2 = (F_1 - F_2) \cup (F_2 - F_1)$.

The comparison model[7] is proposed by Malek and Maeng. In this model, a self-diagnosable system is often represented by a multigraph $M(V,C)$, where V is the same vertex set defined in G and C is the labeled edge set. Let

$(u,v)_w$ be a labeled edge. If (u,v) is an edge labeled by w, then $(u,v)_w$ is said to belong to C, which implies that the vertex u and v are being compared by vertex w. The same pair of vertices may be compared by different comparators, so M is a multigraph. For $(u,v)_w \in C$, we use $r((u,v)_w)$ to denote the result of comparing vertices u and v by w such that $r((u,v)_w)=0$ if the outputs of u and v agree, and $r((u,v)_w)=1$ if the outputs disagree. In this model, if $r((u,v)_w)=0$ and w is fault-free, then both u and v are fault-free. If $r((u,v)_w)=1$, then at least one of the three vertices u, v, w must be faulty. If the comparator w is faulty, then the result of comparison is unreliable that means both $r((u,v)_w)=0$ and $r((u,v)_w)=1$ are possible outputs, and it outputs only one of these two possibilities.

The collection of all comparison results, defined as a function $\sigma: C \rightarrow \{0,1\}$, is called the *syndrome* of the diagnosis. A subset $F \subset V$ is said to be *compatible* with a syndrome σ if σ can arise from the circumstance that all vertices in F are faulty and all vertices in $V–F$ are fault-free. A system is said to be *diagnosable* if, for every syndrome σ, there is a unique $F \subset V$ that is compatible with σ. In [10], a system is called a t-diagnosable system if the system is diagnosable as long as the number of faulty vertices does not exceed t. The maximum number of faulty vertices that the system G can guarantee to identify is called the *diagnosability* of G, written as $t(G)$. Let $\sigma_F=\{\sigma \mid \sigma \text{ is compatible with } F\}$. Two distinct sets $F_1,F_2 \subset V$ are said to be *indistinguishable* if and only if $\sigma_{F1} \cap \sigma_{F2} \neq \emptyset$; otherwise, F_1,F_2 are said to be *distinguishable*. The following theorem given by Sengupta and Dahbura [10] is a necessary and sufficient condition for ensuring distinguishability.

Theorem 1. [10] Let $G(V,E)$ be a graph. For any two distinct sets $F_1,F_2 \subset V$, (F_1,F_2) is a distinguishable pair if and only if at least one of the following conditions is satisfied (see Figure 1):

1. $\exists u,w \in V–F_1–F_2$ and $\exists v \in F_1 \Delta F_2$ such that $(u,v)_w \in C$,
2. $\exists u,v \in F_1–F_2$ and $\exists w \in V–F_1–F_2$ such that $(u,v)_w \in C$, or
3. $\exists u,v \in F_2–F_1$ and $\exists w \in V–F_1–F_2$ such that $(u,v)_w \in C$

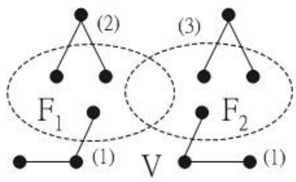

Figure 1: Description of distinguishability for Theorem 1.

An n-dimensional bijective connection network (BC network), denoted by X_n, is an n-regular graph with 2^n vertices and $n2^{n-1}$ edges. The set of all the n-dimensional BC networks is called the family of the n-dimensional BC networks, denoted by L_n. X_n and L_n may be recursively defined as below [4].

Definition 1. The 1-dimensional BC graph X_1 is a complete graph with two vertices. The family of the 1-dimensional BC graph is defined as $L_1 = \{X_1\}$. Let G be a graph. G is an n-dimensional BC graph, denoted by X_n, if there exist V_0, $V_1 \subset V(G)$ such that the following two conditions hold:

1. $V(G) = V_0 \cup V_1$, $V_0 \neq \emptyset$, $V_1 \neq \emptyset$, $V_0 \cap V_1=\emptyset$; and
2. There exists an edge set $M \subset E(G)$ such that M is a perfect matching between V_0 and V_1, $G(V_0) \in L_{n-1}$ and $G(V_1) \in L_{n-1}$.

Before studying the conditional diagnosability of the BC networks, we need some definitions for further discussion. Let $G(V,E)$ be a graph. For any set of vertices $U \subseteq V(G)$, $G[U]$ denotes the subgraph of G induced by the vertex subset U. Let H be a subgraph of G and v be a vertex in H. We use $V(H;3)=\{v \in V(H) \mid deg_H(v) \geq 3\}$ to represent the set of vertices which has degree 3 or more in H. Let $F_1,F_2 \subseteq V(G)$ be two distinct sets and $S=F_1 \cap F_2$. We use $C_{F1\Delta F2,S}$ to denote the subgraph induced by the vertex subset $(F_1 \Delta F_2) \cup \{u \mid$ there exists a vertex $v \in F_1 \Delta F_2$ such that u and v are connected in $G–S\}$. The following result is a useful sufficient condition for checking whether (F_1,F_2) is a distinguishable pair.

Theorem 2. Let $G(V,E)$ be a graph. For any two distinct sets $F_1,F_2 \subset V$ with $|F_i| \leq t$, i=1,2, and $S=F_1 \cap F_2$. (F_1,F_2) is distinguishable if, the subgraph $C_{F1\Delta F2,S}$ of $G–S$ contains at least $2(t\text{-}|S|)+1$ vertices having degree 3 or more.

Proof.

Given any pair of distinct sets of vertices $F_1,F_2 \subset V$ with $|F_i| \leq t$, i=1,2. Let $S=F_1 \cap F_2$, then $0 \leq |S| \leq t–1$, and $|F_1 \Delta F_2| \leq 2(t–|S|)$. Consider the subgraph $C_{F1\Delta F2,S}$, the number of vertices having degree 3 or more is at least $2(t\text{-}|S|)+1$ in $C_{F1\Delta F2,S}$, the subgraph $C_{F1\Delta F2,S}$ contains at least $2(t\text{-}|S|)+1$ vertices. There is at least one vertex with degree 3 or more lying in $C_{F1\Delta F2,S}–F_1\Delta F_2$. Let u be one of such vertices with degree 3 or more. Let i, j, and k be three distinct vertices linked to u. If one of i, j, and k lies in $C_{F1\Delta F2,S}–F_1\Delta F_2$, condition 1 of Theorem 1 holds obviously. Suppose all these three vertices belong to $F_1\Delta F_2$. Without loss of generality, assume i lies in F_1-F_2, one of the two cases will happen: 1) if j lies in F_1-F_2, condition 2 of Theorem 1 holds; or, 2) if j lies in F_2-F_1, wherever k lies in F_1-F_2 or F_2-F_1, condition 2 or 3 of Theorem 1 holds. So (F_1,F_2) is a distinguishable pair and the proof is complete. □

By Theorem 2, we now propose a sufficient condition to verify whether a system is t-diagnosable under the comparison model.

Corollary 1. Let $G(V,E)$ be a graph. G is t-diagnosable if, for each set of vertices $S \subset V$ with $|S| = p$, $0 \le p \le t-1$, every connected component C of $G-S$ contains at least $2(t-p)+1$ vertices having degree at least three. More precisely, $|V(C;3)| \ge 2(t-p)+1$.

3. Conditional Diagnosability of BC Networks X_n

In classical measures of diagnosability for multiprocessor systems under the comparison model, if all the neighbors of some processor v are faulty simultaneously, it is not possible to determine whether processor v is fault-free or faulty. So the diagnosability of a system is limited by its minimum vertex degree.

In an n-dimensional Hypercube Q_n, Q_n has $\binom{2^n}{n}$ vertex subsets of size n, among which there are only 2^n vertex subsets which contains all the neighbors of some vertex. Since the ratio $2^n/\binom{2^n}{n}$ is very small for large n, the probability of a faulty set containing all the neighbors of any vertex is very low. For this reason, Lai et al. introduced a new restricted diagnosability of multiprocessor systems called conditional diagnosability in [5]. They consider the situation that any faulty set cannot contain all the neighbors of any vertex in a system. In the following, we need some terms to define the conditional diagnosability formally. A faulty set $F \subset V$ is called a *conditional faulty set* if $N(v) \not\subset F$ for every vertex $v \in V$. A system $G(V,E)$ is said to be *conditionally t-diagnosable* if F_1 and F_2 are distinguishable, for each pair of conditional faulty sets $F_1, F_2 \subset V$, and $F_1 \ne F_2$, with $|F_1| \le t$ and $|F_2| \le t$. The maximum value of t such that G is conditionally t-diagnosable is called the *conditional diagnosability* of G, written as $t_c(G)$. It is trivial that $t_c(G) \ge t(G)$.

Lemma 1. Let G be a multiprocessor system. Then, $t_c(G) \ge t(G)$.

Now, we give an example to show that the conditional diagnosability of the BC graph X_n is no greater than $3(n-2)+2$, $n \ge 5$. As shown in Figure 2, we take a cycle of length four in X_n. Let $\{v_1,v_2,v_3,v_4\}$ be the four consecutive vertices on this cycle, and let $F_1=N(\{v_1,v_3,v_4\}) \cup \{v_1\}$ and $F_2=N(\{v_1,v_3,v_4\}) \cup \{v_3\}$, then $|F_1|=|F_2|=3(n-2)+2$. It is straightforward to check that F_1 and F_2 are two conditional faulty sets, and F_1 and F_2 are indistinguishable by Theorem 1. Note that the BC graph X_n has no cycle of length 3 and any two vertices have at most two common neighbors. As we can see, $|F_1-F_2|=|F_2-F_1|=1$ and $|F_1 \cap F_2|=3(n-2)+1$. Therefore, X_n is not conditionally $(3(n-2)+2)$-diagnosable and $t_c(X_n) \le 3(n-2)+1$, $n \ge 3$. Then, we shall show that X_n is conditionally t-diagnosable, where $t=3(n-2)+1$.

Lemma 2. $t_c(X_n) \le 3(n-2)+1$ for n $\ge$ 3.

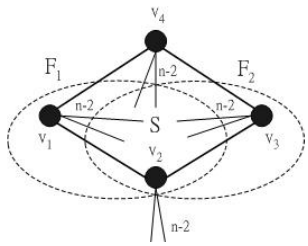

Figure 2: An indistinguishable conditional-pair (F_1,F_2), where $|F_1|=|F_2|=3(n-2)+2$.

Let F be a set of vertices $F \subset V(X_n)$ and C be a connected component of X_n-F. We need some results on the cardinalities of F and $V(C)$ under some restricted conditions. The results are listed in Lemma 3 and 4. In Lemma 3, Zhu proved that deleting at most $2(n-1)-1$ vertices from X_n, the incomplete BC graph X_n has one connected component containing at least $2^n-|F|-1$ vertices. We expand this result further. In Lemma 4, we show that deleting at most $3n-6$ vertices from X_n, the incomplete BC graph X_n has one connected component containing at least $2^n-|F|-2$ vertices.

Lemma 3. [14] $\forall X_n \in L_n$ $(n \ge 3)$, let F be a set of vertices $F \subset V(X_n)$ with $n \le |F| \le 2(n-1)-1$. Suppose that X_n-F is disconnected. Then X_n-F has exactly two components, one is trivial and the other is nontrivial. The nontrivial component of X_n-F contains $2^n-|F|-1$ vertices.

The BC graph can be described as follows: Let X_n denote an n-dimensional BC graph. X_1 is a complete graph with two vertices labeled with 0 and 1, respectively. For $n \ge 2$, each X_n consists of two X_{n-1}'s, denoted by X_{n-1}^L and X_{n-1}^R, with a perfect matching M between them. That is, M is a set of edges connecting the vertices of X_{n-1}^L and the vertices of X_{n-1}^R in a one-to-one manner. It is easy to see that there are 2^{n-1} edges between X_{n-1}^L and X_{n-1}^R. By using a simple induction, we can prove the following lemma.

Lemma 4. $\forall X_n \in L_n$ $(n \ge 5)$, let F be a set of vertices $F \subset V(X_n)$ with $|F| \le 3n-6$. Then X_n-F has a connected component containing at least $2^n-|F|-2$ vertices.

Proof.
We prove the lemma by induction on n. For $n = 5$, it is straightforward to verify that the lemma holds. As the inductive hypothesis, we assume that the result is true for X_{n-1}, for $|F| \le 3(n-1)-6$, and for some $n \ge 6$. Now we consider X_n, $|F| \le 3n-6$. An n-dimensional BC graph X_n can be divided into two X_{n-1}'s, denoted by X_{n-1}^L and X_{n-1}^R. Let $F_L = F \cap V(X_{n-1}^L)$, $0 \le |F_L| \le 3n-6$ and $F_R = F \cap V(X_{n-1}^R)$, $0 \le |F_R| \le 3n-6$. Then $|F| = |F_L| + |F_R|$. Without loss of generality, we may assume that $|F_L| \ge |F_R|$. In the following proof, we consider two cases by the size of F_R: 1) $0 \le |F_R| \le 2$ and 2) $|F_R| \ge 3$.

Case 1: $0 \le |F_R| \le 2$.

Since $0 \leq |F_R| \leq 2$, $X_{n-1}{}^R$-F_R is connected and $|V(X_{n-1}{}^R\text{-}F_R)|=2^{n-1}\text{-}|F_R|$. Let $F_R{}^{(L)} \subset V(X_{n-1}{}^L)$ be the set of vertices which has neighboring vertices in F_R. For each vertex $v \in X_{n-1}{}^L\text{-}F_L\text{-}F_R{}^{(L)}$, there is exactly one vertex $v^{(R)}$ in $X_{n-1}{}^R$-F_R, such that $(v,v^{(R)}) \in E(X_n)$. Besides, $|V(X_{n-1}{}^L\text{-}F_L\text{-}F_R{}^{(L)})| \geq 2^{n-1}\text{-}|F_L|\text{-}|F_R|$. Hence X_n-F has a connected component that contains at least $[2^{n-1}\text{-}|F_R|] + [2^{n-1}\text{-}|F_L|\text{-}|F_R|] = 2^n\text{-}|F|\text{-}|F_R| \geq 2^n\text{-}|F|\text{-}2$ vertices.

Case 2: $|F_R| \geq 3$.

Since $|F_R| \geq 3$, $3 \leq |F_L| \leq 3(n\text{-}1)\text{-}6$ and $3 \leq |F_R| \leq 3(n\text{-}1)\text{-}6$. By the inductive hypothesis, $X_{n-1}{}^L$-F_L ($X_{n-1}{}^R$-F_R, respectively) has a connected component C_L (C_R, respectively) that contains at least $2^{n-1}\text{-}|F_L|\text{-}2$ ($2^{n-1}\text{-}|F_R|\text{-}2$, respectively) vertices. Next, we divide the case into three subcases: 2.1) $|V(C_L)|=2^{n-1}\text{-}|F_L|\text{-}2$ and $X_{n-1}{}^R$-F_R is disconnected, 2.2) $|V(C_L)|=2^{n-1}\text{-}|F_L|\text{-}2$ and $X_{n-1}{}^R$-F_R is connected, and 2.3) $|V(C_L)| \geq 2^{n-1}\text{-}|F_L|\text{-}1$ and $|V(C_R)| \geq 2^{n-1}\text{-}|F_R|\text{-}1$.

Case2.1: $|V(C_L)|=2^{n-1}\text{-}|F_L|\text{-}2$ and $X_{n-1}{}^R$-F_R is disconnected

This is an impossible case. Since $\kappa(X_{n-1})=n\text{-}1$, $|F_R| \geq n\text{-}1$. By Lemma 3, $|F_L| \geq 2((n\text{-}1)\text{-}1)$. Then the total number of faulty vertices is at least $(n\text{-}1) + 2((n\text{-}1)\text{-}1) = 3n\text{-}5$ which is greater than $3n\text{-}6$, a contradiction.

Case 2.2: $|V(C_L)|=2^{n-1}\text{-}|F_L|\text{-}2$ and $X_{n-1}{}^R$-F_R is connected.

Since $X_{n-1}{}^R$-F_R is connected, $|V(X_{n-1}{}^R\text{-}F_R)| = 2^{n-1}\text{-}|F_R|$. Since $|V(C_L)| \geq |F_R| + 1$, there exists a vertex $u \in C_L$ and a vertex $v \in C_R$ such that $(u,v) \in E(X_n)$. Hence X_n-F has a connected component that contains at least $[2^{n-1}\text{-}|F_R|] + [2^{n-1}\text{-}|F_L|\text{-}2] = 2^n\text{-}|F|\text{-}2$ vertices.

Case 2.3: $|V(C_L)| \geq 2^{n-1}\text{-}|F_L|\text{-}1$ and $|V(C_R)| \geq 2^{n-1}\text{-}|F_R|\text{-}1$.

Since $|V(C_L)| \geq |F_R| + 1$, there exists a vertex $u \in C_L$ and a vertex $v \in C_R$ such that $(u,v) \in E(X_n)$. Hence X_n-F has a connected component that contains at least $[2^{n-1}\text{-}|F_L|\text{-}1] + [2^{n-1}\text{-}|F_R|\text{-}1] = 2^n\text{-}|F|\text{-}2$ vertices.

This completes the proof of the lemma. □

By Lemma 4, we have the following corollary.

Corollary 2. $\forall X_n \in L_n$ ($n \geq 5$), let F be a set of vertices $F \subset V(X_n)$ with $|F| \leq 3n\text{-}6$. Then X_n-F satisfies one of the following conditions:

1. X_n-F is connected.
2. X_n-F has two components, one of which is K_1, and the other one has $2^n\text{-}|F|\text{-}1$ vertices.
3. X_n-F has two components, one of which is K_2, and the other one has $2^n\text{-}|F|\text{-}2$ vertices.
4. X_n-F has three components, two of which are K_1, and the third one has $2^n\text{-}|F|\text{-}2$ vertices.

We are now ready to show that the conditional diagnosability of X_n is $3(n–2)+1$ for $n \geq 5$. Let $F_1,F_2 \subset V(X_n)$ be two conditional faulty sets with $F_1 \leq 3(n\text{-}2)+1$ and $F_2 \leq 3(n\text{-}2)+1$, $n \geq 5$. We shall show our result by proving that (F_1,F_2) is a distinguishable conditional-pair under the comparison model.

Lemma 6. Let X_n be an n-dimensional BC graph with $n \geq 5$. For any two conditional faulty sets $F_1,F_2 \subset V(X_n)$, and $F_1 \neq F_2$, with $F_1 \leq 3(n\text{-}2)+1$ and $F_2 \leq 3(n\text{-}2)+1$. Then (F_1,F_2) is a distinguishable conditional-pair under the comparison model.

Proof.

We use Theorem 2 to prove this result. Let $S=F_1 \cap F_2$, then $0 \leq |S| \leq 3(n\text{-}2)$. We will show that, deleting S from X_n, the subgraph $C_{F1\Delta F2,S}$ containing $F_1 \Delta F_2$ has "many" vertices having degree 3 or more. More precisely, we are going to prove that, in the subgraph $C_{F1\Delta F2,S}$ the number of vertices having degree 3 or more is at least $2[3(n\text{-}2)+1\text{-}|S|]+1 = 6n\text{-}2|S|\text{-}9$. In the following proof, we consider three cases by the size of S: 1) $0 \leq |S| \leq n\text{-}1$, 2) $|S|=n$, and 3) $n+1 \leq |S| \leq 3(n\text{-}2)$.

Case 1: $0 \leq |S| \leq n\text{-}1$

Since the connectivity of X_n is n [4], X_n-S is connected, the subgraph $C_{F1\Delta F2,S}$ is the only component in X_n-S. Since the BC graph X_n has no cycle of length three and any two vertices have at most two common neighbors, it is straightforward, though tedious, to check that the number of vertices which has degree 2 or 1 is at most 2 in $C_{F1\Delta F2,S}$. Hence, the number of vertices having degree 3 or more is at least $2^n\text{-}|S|\text{-}2$ which is greater than $6n\text{-}2|S|\text{-}9$, for $n \geq 5$. By Theorem 2, (F_1,F_2) is a distinguishable conditional-pair under the comparison diagnosis model.

Case 2: $|S|=n$

If X_n-S is disconnected, by Lemma 3, X_n-S has one trivial component $\{v\}$ such that $N(v) \subset F_1$ and $N(v) \subset F_2$. Since F_1 and F_2 are two conditional faulty sets, this is an impossible case. So X_n-S is connected, and the subgraph $C_{F1\Delta F2,S}$ is the only component in X_n-S. Let $U=X_n\text{-}(F_1 \cup F_2)$. If there exist two vertices u and v in $V(U)$ such that u is adjacent to v, then the condition 1 of Theorem 1 holds and therefore (F_1,F_2) is a distinguishable conditional-pair; otherwise $V(U)$ is an independent set. Hence, $N_{Xn\text{-}S}(v) \subset F_1 \Delta F_2$, $\forall v \in U$, and we have the following inequality

$$\sum_{v \in U} |\deg_{X_n - S}(v)| \leq \sum_{v \in F_1 \Delta F_2} |\deg_{X_n - S}(v)|.$$

To check the inequality, we have

$$\begin{aligned}&\sum_{v \in U} |\deg_{X_n - S}(v)| \\ &\geq [2^n - 2(3(n-2)+1) + |S|]n - |S|\,n \\ &= n2^n - 6n^2 + 10n\end{aligned}$$

and

$$\sum_{v\in F_1\Delta F_2} |\deg_{X_n-S}(v)| \le 2[3(n-2)+1-|S|]n = 4n^2-10n.$$

$n2^n-6n^2+10n > 4n^2-10n$ for $n\ge 5$, a contradiction.

Case 3: $n+1 \le |S| \le 3(n-2)$

By Corollary 2, there are four cases in X_n-S we need to consider. For case 1 of Corollary 2, X_n-S is connected, the proof is exactly the same as that of Case 2, and hence the detail is omitted. For case 2 and 4 of Corollary 2, X_n-S has at least one trivial component $\{v\}$ such that $N(v)\subset F_1$ and $N(v)\subset F_2$. Since F_1 and F_2 are two conditional faulty sets, the two cases are disregarded. Therefore, we only need to consider that X_n-S has two components, one of which is K_2 and the other one has 2^n-$|S|$-2 vertices. Let (x,y) be the component with only one edge. Since $N(\{x,y\})\subseteq S$ and F_1 and F_2 do not contain all the neighbors of any vertex, vertex x and y cannot belong to $F_1\Delta F_2$. So the subgraph $C_{F1\Delta F2,S}$ is the other large connected component of X_n-S. Let $U=X_n$-$(F_1\cup F_2)$-$\{x,y\}$. If there exist two vertices u and v in $V(U)$ such that u is adjacent to v, then the condition 1 of Theorem 1 holds and therefore (F_1,F_2) is a distinguishable conditional-pair; otherwise $V(U)$ is an independent set. Hence, $N_{Xn-S}(v)\subset F_1\Delta F_2$, $\forall v\in U$, and we have the following inequality

$$\sum_{v\in U}|\deg_{X_n-S}(v)| \le \sum_{v\in F_1\Delta F_2}|\deg_{X_n-S}(v)|.$$

To check the inequality, we have

$$\sum_{v\in U}|\deg_{X_n-S}(v)| \ge [2^n-2(3(n-2)+1)+|S|-2]n-|S|n = n2^n-6n^2+8n$$

and

$$\sum_{v\in F_1\Delta F_2}|\deg_{X_n-S}(v)| \le 2[3(n-2)+1-|S|]n \le 4n^2-12n.$$

$n2^n-6n^2+8n > 4n^2-12n$ for $n\ge 5$, a contradiction.

In Case 1, we prove that at least one of the conditions of Theorem 1 is satisfied in subgraph $C_{F1\Delta F2,S}$. In Case 2 and 3, the condition 1 of Theorem 1 holds in subgraph $C_{F1\Delta F2,S}$. Therefore, (F_1,F_2) is a distinguishable conditional-pair under the comparison model. □

By Lemma 2, $t_c(X_n) \le 3(n-2) + 1$, and by Lemma 6, X_n is conditionally $(3(n-2)+1)$-diagnosable for $n \ge 5$. We now present our main result which can be stated as follows.

Theorem 4. The conditional diagnosability of X_n is $t_c(X_n)=3(n-2)+1$ for $n \ge 5$.

Since Q_n, CQ_n, TQ_n, $MQ_n \in L_n$, the following corollary holds.

Corollary 3. $t_c(Q_n) = t_c(CQ_n) = t_c(TQ_n) = t_c(MQ_n) = 3(n-2)+1$ for $n \ge 5$.

4. Conclusions

In the real world, processors fail independently and with different probabilities. The probability that any faulty set contains all the neighbors of some processor is very small[1],[8], so we are interested in the study of conditional diagnosability. A new diagnosis measure proposed by Lai et al.[5], it restricts that each processor of a system is incident with at least one fault-free processor. In this paper, we use the BC graph as an example and show that the conditional diagnosability of X_n is $3(n-2)+1$ under the comparison model.

Several different fault diagnosis models have gained much attention in the study of fault diagnosis. It is worth to investigate the conditional diagnosability of a system under various models. It is also an attractive work to develop more different measures of diagnosability based on network topology and network reliability.

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