Strong Menger Connectivity on the Bubble-sort Graphs

Chieh-Feng Chiang and Jimmy J. M. Tan
{cfchiang, jmtan}@ cs.nctu.edu.tw
Department of Computer Science, National Chiao Tung University, Hsinchu, Taiwan

Abstract

Interconnection networks have been widely discussed in recent days, and usually represented as undirected graphs. One of the most important properties of graphs is the connectivity, which is defined as the minimum number of vertices removed from a graph resulting in a disconnected graph or a graph containing a single isolated vertex. A famous theorem by Menger [4] pointed out the equivalence of the minimum size of an x, y-cut and the maximum number of pairwise internally disjoint x, y-paths for some nonadjacent vertices x, y in a graph. After that, Oh and Chen [5] extended this concept to introduce the property called strongly Menger connectivity, which describes that, with some faulty vertices, the number of x, y-paths is the minimum remaining degree of them. In this paper, we focus on a recursively constructed interconnection network, the bubble-sort graph. We prove that in an n-dimensional bubble-sort graph B_n with at most $n-3$ faulty vertices, it is strongly Menger-connected.

1 Introduction

With the rapid development of technology, interconnection networks have been widely discussed in recent days. As in customary, we view the architecture of the underlying interconnection networks as graphs. An undirected graph $G = (V, E)$ is usually used representing a network, where $V(G)$ stands for the set of all processors and $E(G)$ stands for the connecting links between the processors.

The connectivity is an important property of interconnection networks. A subset S of vertices $V(G)$ in a graph G is a *cut set* if the induced subgraph $G-S$ is disconnected. The *connectivity* of G is defined as the minimum size of a vertex cut if G is not a complete graph, and that is defined as the number of vertices minus one if otherwise. We say that a graph G is *k-connected* if k is not larger than its connectivity.

Following the concept of connectivity, a classical theorem was proposed by Menger stating the relationship of an x, y-cut and the number of pairwise internally disjoint x, y-paths for some nonadjacent vertices x, y in a graph G. The usefulness of this Menger-connectivity has been stated on some applications. In 2004, Dekker et al. [3] stressed the relationship between vertex connectivity and network symmetry, along with a network design and analysis tool called CAVALIER, to assist with the process of designing robust networks. In 2006, Peserico et al. [6] gave a rigorous formalization of the intuitive notion of "hole" in a graph, and characterizes networks where connectivity depends on the "big picture" structure of the network, and not on the local "noise" caused by faulty or imprecisely positioned vertices and links.

Based on the basic definition of Menger-connectivity, Oh and Chen gave an enhancement of it, which was named as the *strong Menger connectivity.*

Theorem 1. *[4] Let x and y be two distinct vertices of a graph G and $(x, y) \notin E(G)$. The minimum size of an x, y-cut equals the maximum number of pairwise internally disjoint x, y-paths.*

Definition 1. *[5] A k-regular graph G is strongly Menger-connected if for any subgraph $G-F$ of G with at most $k-2$ vertices removed, each pair of vertices u and v in $G-F$ are connected by $min\{deg_{G-F}(u), deg_{G-F}(v)\}$ vertex-disjoint fault-free paths in $G-F$, where $deg_{G-F}(u)$ and $deg_{G-F}(v)$ are the degree of u and v in $G-F$, respectively.*

In order to be consistent with Definition 1, we say that a graph G possess the strongly

Menger-connected property with respect to a vertex set F if, after deleting F from G, there are $min\{deg_{G-F}(u), deg_{G-F}(v)\}$ vertex-disjoint fault-free paths connecting u and v, for each pair of vertices u and v in $G-F$. We also call a graph "strongly Menger-connected", and omit the description of the remaining structure $G-F$ of the graph, if there is no ambiguity.

Among all well-known topologies, the bubble-sort graphs, which belong to the class of Cayley graphs generated by a transposition tree, have been attractive alternative to the famous hypercubes. They have some good topological properties such as highly symmetry and recursive structure. The n-dimensional bubble-sort graph B_n has $n!$ vertices, and is vertex transitive. The connectivity of B_n is $n-1$ and the diameter is $\frac{n(n-1)}{2}$. The formal definition of bubble-sort graphs is stated below, and the example of the bubble-sort graphs of dimension 2, 3, and 4 are shown in Fig. 1.

Definition 2. *[1] The n-dimensional bubble-sort graph B_n has vertex set that consists of all $n!$ permutations on $\{1, 2, ..., n\}$. A permutation x on $\{1, 2, ..., n\}$ is denoted as $x = x_1x_2...x_n$. A vertex x is adjacent to $x^{(i)} = x_1...x_{i-1}x_{i+1}x_ix_{i+2}...x_n$ for all $1 \leq i \leq n-1$.*

In this paper, we extend the connectivity result of B_n further, which has been proved that the connectivity of an n-dimensional bubble-sort graph B_n is $n-1$. We shall study the strongly Menger-connected property of B_n with at most $n-3$ vertices deleted. That is, an n-dimensional bubble-sort graph B_n with at most $n-3$ faulty vertices is strongly Menger-connected.

2 Preliminaries

In the next section, we are going to prove that the n-dimensional bubble-sort graphs B_n are strongly Menger-connected if there are at most $n-3$ faulty vertices. Before proving this main result, we need a critical lemma, which implies that every n-dimensional bubble-sort graph with no more than $2n-5$ vertex faults, still contains a large connected component.

A recent research paper discussed what happens when the number of faults in the Cayley graph generated by a transposition tree is linear in the degree. Note that the set of bubble-sort graphs is a subset of

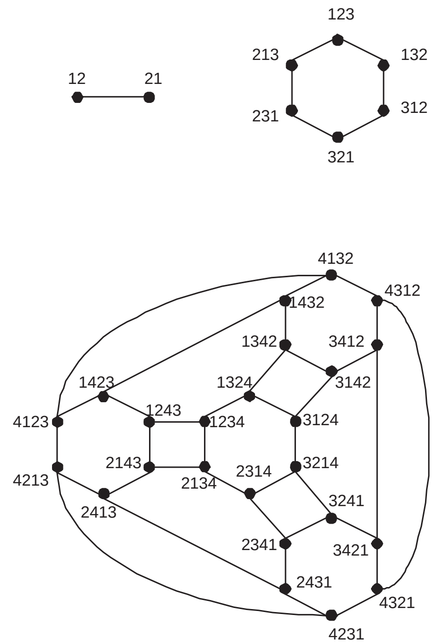

Figure 1: The bubble-sort graphs B_2, B_3, and B_4.

the Cayley graphs generated by a transposition tree, that is, the bubble-sort graphs are special cases of Cayley graph. The disconnection status of removing some vertices is shown in the next theorem.

Theorem 2. *[2] Let $\Gamma_n(S)$ be a Cayley graph generated by a transposition tree $G(S)$. If T is a set of vertices with $|T| \leq k(n-1) - \frac{k(k+1)}{2}$, where $1 \leq k \leq n-2$, then $\Gamma_n(S) - T$ has one large (connected) component, and the remaining small components have at most $k-1$ vertices in total.*

If the number k equals 2, we can rewrite this theorem with respect to the terminologies of bubble-sort graphs as the next lemma.

Lemma 1. *Let B_n be an n-dimensional bubble-sort graph, and F be a set of vertices in B_n with $|F| \leq 2n-5$. Either the induced subgraph $B_n - F$ is connected, or $B_n - F$ has two components one of which is an isolated vertex.*

Proof. This lemma follows trivially from Theorem 2 when $k = 2$. □

By replacing a faulty vertex by a faulty edge, a similar result is shown below.

Lemma 2. *Let B_n be an n-dimensional bubble-sort graph, e_f be an edge in B_n, and F_v be a set of vertices* *in B_n with $|F_v| \leq 2n-6$. Either the induced subgraph $B_n - F_v - \{e_f\}$ is connected, or $B_n - F_v - \{e_f\}$ has two components one of which is an isolated vertex.*

Proof. Since $|F_v| \leq 2n-6$, the induced subgraph $B_n - F_v$ is either connected or with two components one of which is an isolated vertex, according to Lemma 1 ($2n-6 < 2n-5$). Now we consider the two situations step by step, and check the result after deleting the faulty edge e_f in graph $B_n - F_v$.

For the first situation, suppose $B_n - F_v$ has two connected components, and one of which is a single isolated vertex. As we know, deleting a vertex from a graph generates less number of edges than that of deleting an edge from a graph. So deleting one end of edge e_f from graph $B_n - F_v$ will result in a subgraph of less number of edges than just deleting edge e_f. Let $e_f = (u, v)$. Arbitrarily choose one end of e_f, without loss of generality the vertex u, we get the result $|F_v \cup \{u\}| \leq 2n-5$. By Lemma 1, the induced subgraph $B_n - F_v - \{u\}$ also contain one connected component and one isolated vertex. So does the graph $B_n - F_v - \{e_f\}$.

For the second situation, suppose $B_n - F_v$ is connected. If $B_n - F_v - \{e_f\}$ is connected, the proof is completed. Otherwise, let C_u, C_v be two components of $B_n - F_v - \{e_f\}$, and $e_f = (u, v)$ where $u \in C_u$, $v \in C_v$. Without loss of generality, assume $|V(C_u)| \leq |V(C_v)|$. Since $|F_v \cup \{v\}| \leq 2n-5$, either the subgraph $B_n - F_v - \{v\}$ is connected, or it contains one connected component and one isolated vertex, according to Lemma 1. Because the number of edges in $B_n - F_v - \{v\}$ is less than that of $B_n - F_v - \{e_f\}$, either $B_n - F_v - \{e_f\}$ is connected or $B_n - F_v - \{e_f\}$ has two components, one of which is a single isolated vertex. □

So far, the preparation for our main theorem has been set up. In the next section, the strong Menger connectivity of bubble-sort graphs will be retrieved consequently.

3 Strong Menger Connectivity on the Bubble-sort Graphs

Theorem 3. *Let B_n be an n-dimensional bubble-sort graph, and let F be a set of faulty vertices with $|F| \leq n-3$. Each pair of vertices u and v in $B_n - F$ are connected by $min\{deg_{B_n-F}(u), deg_{B_n-F}(v)\}$ vertex-disjoint fault-free paths, where $deg_{B_n-F}(u)$ and $deg_{B_n-F}(v)$ are the remaining degrees of u and v in $B_n - F$, respectively.*

Proof. Let u and v be two fault-free vertices in $B_n - F$, where F is a set of faulty vertices with $|F| \leq n-3$. Firstly, assume without loss of generality that $deg_{B_n-F}(u) \leq deg_{B_n-F}(v)$, then $min\{deg_{B_n-F}(u), deg_{B_n-F}(v)\} = deg_{B_n-F}(u)$. In this proof, we will show that u is connected to v if the number of vertices deleted is smaller than $deg_{B_n-F}(u) - 1$ in $B_n - F$. By Theorem 1, this implies that each pair of vertices u and v in $B_n - F$ are connected by $min\{deg_{B_n-F}(u), deg_{B_n-F}(v)\}$ vertex-disjoint fault-free paths, where $|F| \leq n-3$.

There are two situations we must consider: either vertices u and v are nonadjacent or they are adjacent.

Considering the first situation, vertices u and v are not adjacent. For the sake of contradiction, suppose that u and v are separated by deleting a set of vertices V_f, where $|V_f| \leq deg_{B_n-F}(u) - 1$. As a consequence, $|V_f| \leq n-2$ because that $deg_{B_n-F}(u) \leq deg(u) \leq n-1$. Summing up the cardinality of these two sets F and V_f we get $|F| + |V_f| \leq 2n-5$. Let $S = F \cup V_f$. By Lemma 1, it implies that in subgraph $B_n - S$, (**i**) either there is only one connected component, or (**ii**) there are two components, one of which contains only one isolated vertex. If $B_n - S$ is connected, it contradicts to the assumption that u and v are disconnected. Otherwise, if $B_n - S$ has two components and one of which contains only one vertex x. Since we assume that u and v are separated, one of u and v is the vertex x, say $u = x$. Thus, the set V_f must be the neighborhood of u and $|V_f| = deg_{B_n-F}(u)$, which is also a contradiction. (Remind that actually $|V_f| \leq deg_{B_n-F}(u) - 1$.) Then, u is connected to v when the number of vertices deleted is smaller than $deg_{B_n-F}(u) - 1$ in $B_n - F$.

Considering the second situation, vertices u and v are adjacent. We need to show that u is connected to v if the number of vertices deleted is smaller than $deg_{B_n-F}(u) - 2$ in $B_n - F - \{(u, v)\}$. Suppose on the contrary that in $B_n - F - \{(u, v)\}$, u and v are separated by deleting a set of vertices V_f, where $|V_f| \leq deg_{B_n-F}(u) - 2$. Since $deg_{B_n-F}(u) \leq n-1$, we get $|V_f| \leq n-3$. Then the union set S of F and V_f has cardinality $|S| = |F| + |V_f| \leq 2n-4$. In this circumstance, Lemma 2 implies either that (**i**) $B_n - S - \{(u, v)\}$ is connected or that (**ii**) $B_n - S - \{(u, v)\}$ has two components one of which is an isolated vertex. For the first situation, that $B_n - S - \{(u, v)\}$ is connected is a contradiction to the assumption that u and v are separated in $B_n - F - \{(u,v)\}$. For the second situation, the single isolated vertex must be u or v, and we without loss of generality let u be such vertex. In $B_n - F - \{(u,v)\}$, in order the make u an isolated vertex, the number of vertices deleted must be greater or equal to $deg_{B_n-F}(u) - 1 = n-2$. However, $|V_f| \leq n-3$, which is a contradiction. So u and v are connected when the number of vertices deleted is smaller than $deg_{B_n-F}(u) - 2$ in $B_n - F - \{(u,v)\}$.

The proof is complete. □

As a short conclusion, we have proved that in an n-dimensional bubble-sort graph B_n with a set of faulty vertices F where $|F| \leq n-3$, the number of vertex-disjoint fault-free paths between each pair of vertices u and v is the minimum remaining degree of them. In other words, bubble-sort graphs are proved to be strongly Menger connected.

Acknowledgment

The authors would like to thank the anonymous referees for their valuable comments and suggestions that improve the quality of this paper.

This work was supported in part by the National Science Council of the Republic of China under Contract NSC 96-2221-E-009-137-MY3, and in part by the Aiming for the Top University and Elite Research Center Development Plan.

References

[1] S. B. Akers and B. Krishnamurthy “A group-theoretic model for symmetric interconnection networks,” *IEEE Trans. Computers*, vol. 38, pp. 555-566, 1989.

[2] E. Chen, ”Linearly many faults in Cayley graphs generated by transposition trees,” *Information Sciences*, vol. 177, pp. 4877-4882, 2007.

[3] A. H. Dekker and B. D. Colbert, ”Network Robustness and Graph Topology,” *Proceedings of the 27th Australasian Conference on Computer Science*, vol. 26, pp. 359-368, January 2004.

[4] K. Menger, ”Zur allgemeinen kurventheorie,”*Fund. Math*, vol. 10, pp. 95-115, 1927.

[5] E. Oh and J. Chen, “On strong Menger-connectivity of star graphs,” *Discrete Applied Mathematics*, vol. 129, pp. 499-511, 2003.

[6] E. Peserico and L. Rudolph, “Robust Network Connectivity: When It’s the Big Picture That Matters,” *ACM SIGMETRICS Performance Evaluation Review*, vol. 34, pp. 299-310, June 2006.

[7] L.-M. Shih, C.-F. Chiang, L.-H. Hsu, and J. J.-M. Tan, “Strong Menger Connectivity with Conditional Faults on the Class of Hypercube-like Networks,” *Information Processing Letters*, vol. 106, pp. 64-69, April 2008.