Maximally m-induced Subgraph of Some Interconnection Networks

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Abstract

The topological structure of an interconnection network can be modeled by a graph $G = (V, E)$ where V is the vertex set and E the edge set of G. For a vertex subset $V' \subseteq V$ of graph G, the subgraph of G induced by V', denoted by $G[V']$, is a graph with vertex set V' and all the edges of G with both ends of vertices in V'. An m-induced subgraph of a graph is such one which induced by m vertices. A maximally m-induced subgraph of a graph G, denoted by $V_m^{\max}(G)$, can be defined as $V_m^{\max}(G) = \{G[V'] \mid \max_{V' \subseteq V, |V'|=m} |E(G[V'])|\}$. Let $\max_m(G)$ be the number of edges in such a maximally m-induced subgraph $V_m^{\max}(G)$. Let m be an integer with $m = \sum_{i=0}^{r-1} 2^{l_i}$ and $l_0 > l_1 > \cdots > l_{r-1}$. $g(m) = \sum_{i=0}^{r-1}(\frac{l_i}{2} + i)2^{l_i}$. For an n-dimensional hypercube Q_n, it is proved by Abdel-Ghaffar in 2003 that $\max_m(Q_n) = g(m)$ for $n \geq 1$ and $0 \leq m \leq 2^n$. In this paper, we investigate in the maximally m-induced subgraph of the generalized hypercubes GQ_n and show that $\max_m(GQ_n) = g(m)$ for $n \geq 3$ and $0 \leq m \leq 2^n$. The hypercubes, twisted cubes, crossed cubes, and möbius cubes are special cases of generalized hypercubes. Additionally, we provide an algorithm to find a maximally m-induced subgraph of generalized hypercubes.

Keywords: subgraph, hypercube, generalized hypercube, maximally m-induced subgraph.

1 Introduction

The topological structure of an *interconnection network* can be modeled by a *graph*, while vertices represent *processors* and edges represent *links* between processors. For the purpose of connecting hundreds or thousands of processing elements, many interconnection network topologies have been proposed in literature. Graph theory can be used to analyze the networks and most of the graph definitions we use are standard [4]. Terms networks and graphs are used interchangeably in this paper.

Given a graph $G = (V, E)$ where V is the vertex set and E the edge set of G. For a vertex subset V' of graph G, the subgraph of G induced by V', denoted by $G[V']$, is a graph with vertex set V' and all the edges of G with both ends of vertices in V'. An *m-induced subgraph* of a graph is such one which induced by m vertices [16]. A *maximally m-induced subgraph* of a graph G, denoted by $V_m^{\max}(G)$, can be defined as

$$V_m^{\max}(G) = \{G[V'] \mid \max_{V' \subseteq V, |V'|=m} |E(G[V'])|\}.$$

Let $\max_m(G)$ be the number of edges in such a maximally m-induced subgraph $V_m^{\max}(G)$. The *n-dimensional hypercube* [3], denoted by Q_n, is an undirected graph with 2^n vertices, which consists of all n-bit binary strings as its vertices. Take Q_3 for an instance, let V_1 be the vertex set $\{000, 001, 011, 111\}$, then $E(Q_3[V_1]) = \{(000, 001), (001, 011), (011, 111)\}$. However, let V_2 be the vertex set $\{000, 001, 011, 010\}$, then $E(Q_3[V_2]) = \{(000, 001), (001, 011), (011, 010), (010, 000)\}$. Actually, $\max_4(Q_3) = 4$. To maximize the number of edges joining vertices of a vertex set with m vertices of a graph is an important issue in this research.

Let $m = \sum_{i=0}^{r-1} 2^{l_i}$, where $l_0 > l_1 > \cdots > l_{r-1} \geq 0$. For an n-dimensional hypercube Q_n, it is proved in [1] that $\max_m(Q_n) = \sum_{i=0}^{r-1}(\frac{l_i}{2} + i)2^{l_i}$. For the *recursive circulant graphs* $RC(2^n, 4)$, the value of $\max_m(RC(2^n, 4))$ is proved in [16]. The results of maximally m-induced subgraph have applications in the evaluation of fault tolerance and bandwidth of networks. Maximizing the number of transitions corresponding to single edges may decrease the power consumption because of switching activities in processors [14]. In addition, they also relate to electro-

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mechanical or optical sensors [16]. The same technique also can be used to facilitate browsing of documents in libraries and data storage systems [12]. Some other applications can be seen in [1].

The n-dimensional hypercube, denoted by Q_n, is a popular network because of its attractive properties, including *regularity*, *symmetry*, *small diameter*, *strong connectivity*, *recursive construction*, *partitionability*, and *relatively low link complexity* [3, 11, 13]. There are some variations of the hypercube Q_n appearing in literature, such as *twisted cubes* [2, 10], *crossed cubes* [9, 15], and *möbius cubes* [8, 15]. These variations preserve most of the good topological properties of the hypercube, and even better. For example, the diameter of these variation cubes is around half of that of the hypercube. Recently, the twisted cubes, crossed cubes, and möbius cubes are proved to be super connected and super fault-tolerant hamiltonian graphs [5, 7]. We define a generalization of those graphs. The n-dimensional *generalized hypercubes*, denoted by GQ_n, are generalizations of the the n-dimensional twisted cubes TQ_n, crossed cubes CQ_n, and möbius cubes MQ_n. Let r and $l_0 > l_1 > \cdots > l_{r-1}$ be nonnegative integers with $m = \sum_{i=0}^{r-1} 2^{l_i}$. In this paper, we show that $\max_m(GQ_n) = \sum_{i=0}^{r-1}(\frac{l_i}{2} + i)2^{l_i}$ for $n \geq 3$ and $0 \leq m \leq 2^n$. Moreover, we provide an algorithm to find the maximally m-induced subgraph $V_m^{max}(GQ_n)$ of graph GQ_n for $n \geq 3$ and $0 \leq m \leq 2^n$.

The rest of this paper is organized as follows. Section 2 starts with the definition of generalized hypercubes and defines the function $g(m)$. Section 3 forms the main result of the paper. In Section 4, we give the conclusion remarks.

2 Preliminary

Motivated by the recursively structure of the hypercubes, crossed cubes, twisted cubes, and möbius cubes, we have the following n-dimensional generalized hypercubes, denoted by GQ_n. The GQ_n for $n \geq 0$ is recursively defined as follows. For $n = 0$, GQ_0 is a vertex. For $n = 1$, GQ_1 is isomorphic to the 1-dimensional hypercube Q_1 with vertex set $\{0, 1\}$ and edge set $\{(0, 1)\}$. As for $n \geq 2$, GQ_n consists of (1) two not necessarily identical GQ_{n-1}'s, denoted by GQ_{n-1}^0 and GQ_{n-1}^1; and (2) an arbitrary perfect matching with 2^{n-1} edges between the two GQ_{n-1}'s, each vertex in GQ_{n-1}^0 is adjacent to exactly one vertex in GQ_{n-1}^1. The n-dimensional generalized hypercubes GQ_n for $n = 1, 2$ are shown in Figure 1, in which the edge set of GQ_2 has two different situations. Figure 2 illustrates labels of the vertex set of GQ_n.

Now, the Function $g(m)$ is defined as the following. Let m be an integer with $m = \sum_{i=0}^{r-1} 2^{l_i}$ and $l_0 > l_1 > \cdots > l_{r-1}$. Then, $g(m) = \sum_{i=0}^{r-1}(\frac{l_i}{2} + i)2^{l_i}$. As an example, for $n = 86 = 2^6 + 2^4 + 2^2 + 2^1$, $g(86) = (6/2+0)2^6 + (4/2+1)2^4 + (2/2+2)2^2 + (1/2+3)2^1 = 259$.

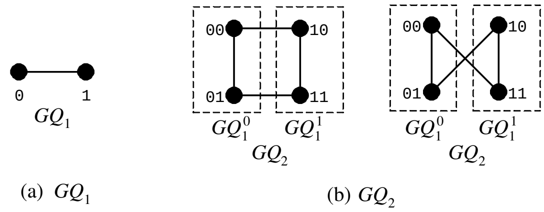

Figure 1. (a) GQ_1; (b) Two situations of GQ_2.

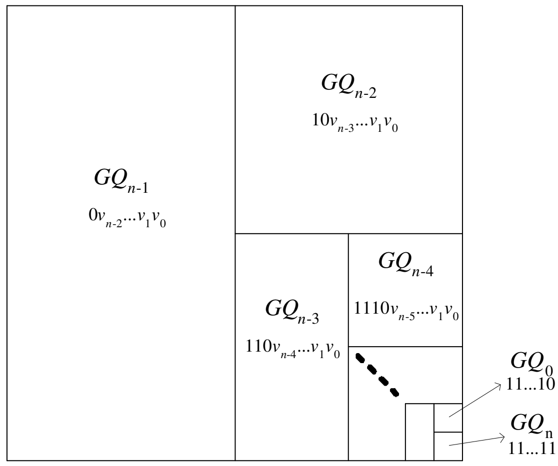

Figure 2. Labels of the vertex set of GQ_n.

3 Maximally m-induced Subgraph of Generalized Hypercubes

In this section, we state and show the main result that given a generalized hypercube GQ_n for $n \geq 3$ and an integer m for $0 \leq m \leq 2^n$, we have that $\max_m(GQ_n) = g(m)$. In order to prove it, the following lemma is needed.

Lemma 1 *[16] For any nonnegative integers m_0,m_1, $g(m_0 + m_1) \geq g(m_0) + g(m_1) + \min\{m_0, m_1\}$.*

The following lemma shows that for $n \geq 3$ and $0 \leq m \leq 2^n$, the maximally m-induced subgraph $V_m^{\max}(GQ_n)$ of GQ_n contains at most $g(m)$ edges by induction.

Lemma 2 *Given a generalized hypercube GQ_n for $n \geq 3$, and an integer m for $0 \leq m \leq 2^n$. We have that $\max_m(GQ_n) \leq g(m)$.*

Proof. This lemma is proved by induction. For the induction base $n = 3$, it is not hard to check that $\max_m(GQ_3) \leq g(m)$ for $0 \leq m \leq 8$ by brute force. Assume that $\max_m(GQ_n) \leq g(m)$ for $0 \leq m \leq 2^n$. Now, we shall show that for the m-induced subgraph of the GQ_{n+1},

$\max_m(GQ_{n+1}) \leq g(m)$ for $0 \leq m \leq 2^{n+1}$. In the m-induced subgraph with m vertices of GQ_{n+1}, we may assume that there are m_0 vertices in GQ_n^0 and m_1 in GQ_n^1 with $m = m_0 + m_1$. Without loss of generality, we may assume that $m_0 \geq m_1 \geq 0$. We divide the proof into the following two cases.

Case 1: $m_1 = 0$**.** So the m vertices are all distributed in GQ_n^0 and $m \leq 2^n$. By the induction hypothesis, we have that $\max_m(GQ_{n+1}) \leq g(m)$.

Case 2: $m_1 > 0$**.** For the maximally m-induced subgraph of GQ_{n+1}, there are $m_0 > 0$ vertices in GQ_n^0 and m_1 in GQ_n^1. Hence, $\max_m(GQ_{n+1}) \leq \max_{m_0}(GQ_n^0) + \max_{m_1}(GQ_n^1) + \min\{m_0, m_1\}$. By the induction hypothesis, $\max_{m_0}(GQ_n^0) \leq g(m_0)$ and $\max_{m_1}(GQ_n^1) \leq g(m_1)$. In addition, $g(m_0)+g(m_1)+\min\{m_0, m_1\} \leq g(m_0+m_1)$ by Lemma 1. As a result, we have the following equation and this lemma is proved.

$$\begin{aligned} \max_m(GQ_{n+1}) &\leq \max_{m_0}(GQ_n^0) + \max_{m_1}(GQ_n^1) \\ &\quad + \min\{m_0, m_1\} \\ &\leq g(m_0) + g(m_1) + \min\{m_0, m_1\} \\ &\leq g(m_0 + m_1) \\ &= g(m). \end{aligned}$$ $\diamond$

Now, we give an algorithm to find an m-induced subgraph of GQ_n with $g(m)$ edges.

Algorithm

01. $V' :=$ BUILD_VERTEX_SET(n, m);
 /* Give an n-dimensional generalized GQ_n for $n \geq 3$, and an integer m, where $0 \leq m \leq 2^n$. Let $m = \sum_{i=0}^{r-1} 2^{l_i}$, where $l_0 > l_1 > \cdots > 0 l_{r-1} \geq 0$. */
02. BUILD_VERTEX_SET(n, m)
03. begin
04. $V' := \emptyset$; /* V' is a vertex subset of GQ_n. */
05. if $(m = 0)$ then return $V' := \emptyset$;
06. if $(m = 2^n)$ then return $V' := V(GQ_n)$;
07. for $i := 0$ to $r - 1$
08. $V' := V' \cup \{1^{n-(l_i+1)} 0 v_{l_i-1} \cdots v_1 v_0 | v_x \in \{0, 1\}$ for $0 \leq x \leq l_i - 1\}$;
09. return V'; /* $GQ_n[V']$ is the maximally m-induced subgraph of GQ_n */
10. end BUILD_VERTEX_SET

Take one situation of GQ_3 as Figure 3 for example, while $m = 7$, $V' = \{000, 001, 010, 011, 100, 101, 110\}$. Now, we investigate in the number of edges of $GQ_n[V']$ of the above algorithm. Firstly, if $m = 0$, by Line 5 of the algorithm, $V' = \emptyset$ and $|E(GQ_n[V'])| = 0 = g(0)$. Secondly, if $m = 2^n$, by Line 6 of the algorithm, $V' = V(GQ_n)$ and $|E(GQ_n[V'])| = 2^n \times \frac{n}{2} = g(2^n)$. Finally, we consider that $0 < m < 2^n$. Let $m = \sum_{i=0}^{r-1} 2^{l_i}$, where $l_0 > l_1 > \cdots > l_{r-1} \geq 0$. After finishing the for loop of the algorithm (lines 7-8), Table 1 is established, and the total number of edges of $GQ_n[V']$ is $g(m)$. Therefore, Lemma 3 follows.

Table 1. Total number of edges of $GQ_n[V']$ with $0 < m < 2^n$.

for loop	$\|E(GQ_n[V'])\|$
$i = 0$	$\|E(GQ_{l_0})\|$
$i = 1$	$\|E(GQ_{l_0})\| + (\|E(GQ_{l_1})\| + 2^{l_1})$
$i = 2$	$\|E(GQ_{l_0})\| + (\|E(GQ_{l_1})\| + 2^{l_1})$ $+ (\|E(GQ_{l_2})\| + 2 \times 2^{l_2})$
$i = 3$	$\|E(GQ_{l_0})\| + (\|E(GQ_{l_1})\| + 2^{l_1})$ $+ (\|E(GQ_{l_2})\| + 2 \times 2^{l_2})$ $+ (\|E(GQ_{l_3})\| + 3 \times 2^{l_3})$
$\cdots$	$\cdots$
$i = r - 1$	$\|E(GQ_{l_0})\| + (\|E(GQ_{l_1})\| + 2^{l_1})$ $+ (\|E(GQ_{l_2})\| + 2 \times 2^{l_2})$ $+ (\|E(GQ_{l_3})\| + 3 \times 2^{l_3})$ $+ \cdots$ $+ (\|E(GQ_{l_{r-1}})\| + (r-1) \times 2^{l_{r-1}})$ $= 2^{l_0} \times \frac{l_0}{2} + (2^{l_1} \times \frac{l_1}{2} + 2^{l_1})$ $+ (2^{l_2} \times \frac{l_2}{2} + 2 \times 2^{l_2})$ $+ (2^{l_3} \times \frac{l_3}{2} + 3 \times 2^{l_3})$ $+ \cdots$ $+ (2^{l_{r-1}} \times \frac{l_{r-1}}{2} + (r-1) \times 2^{l_{r-1}})$ $= g(m)$

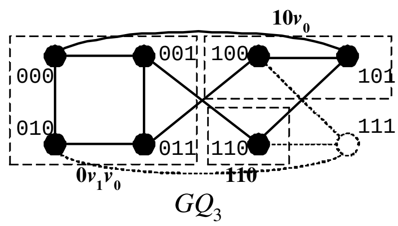

Figure 3. Maximally m-induced subgraph of the generalized hypercube GQ_3 with $m = 7$.

Lemma 3 *Given a generalized hypercube GQ_n for $n \geq 3$, and an integer m for $0 \leq m \leq 2^n$. We have that $\max_m(GQ_n) \geq g(m)$.*

According to Lemma 2 and Lemma 3, the main result of this paper is stated as Theorem 1.

Theorem 1 *Given a generalized hypercube GQ_n for $n \geq 3$, and an integer m for $0 \leq m \leq 2^n$. We have that $\max_m(GQ_n) = g(m)$.*

By the construction scheme of generalized hypercubes, the hypercubes, crossed cubes , twisted cubes, and möbius cubes are special cases of generalized hypercubes. As a result, we have the following corollary.

Corollary 1 $\max_m(Q_n) = \max_m(CQ_n) = \max_m(TQ_n) = \max_m(MQ_n) = g(m)$ *for* $n \geq 3$ *and* $0 \leq m \leq 2^n$.

4 Conclusion Remarks

The n-dimensional generalized hypercube GQ_n is a promising candidate for interconnection networks. Additionally, the crossed cubes CQ_n, twisted cubes TQ_n, and möbius cubes MQ_n are special cases of the GQ_n. This research determined the maximum number of edges of a subgraph of the GQ_n induced by a given number of m vertices with $0 \leq m \leq 2^n$. We also give an algorithm to find the maximally m-induced subgraph $V_m^{\max}(GQ_n)$ of generalized hypercubes.

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