# Spider Web Networks

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## Abstract

In this paper, we propose a variation of honeycomb meshes, called spider web networks. Assume that $m$ and $n$ are positive even integers with $m \geq 4$. A spider web network $SW(m,n)$ is a 3-regular bipartite planar graph with bipartition $C$ and $D$. We prove that the honeycomb rectangular mesh $HREM(m,n)$ is a spanning subgraph of $SW(m,n)$. We also prove that $SW(m,n) - e$ is hamiltonian for any $e \in E$ and $SW(m,n) - \{c,d\}$ remains hamiltonian for any $c \in C$ and $d \in D$. These hamiltonian properties are optimal.

## 1 Introduction

Throughout this paper, we assume that $m, n$ are positive even integers with $m \geq 4$. We use $[r]_s$ to denote $r$ ($\mathbf{mod}\ s$).

Network topology is a crucial factor for an interconnection network since it determines the performance of the network. Many interconnection network topologies have been proposed in the literature for the purpose of connecting a large number of processing elements. Network topology is always represented by a graph where nodes represent processors and edges represent links between processors. One of the most popular architecture is the mesh connected computers [7]. Each processor is placed in a square or rectangular grid and is connected by a communication link to its neighbors up to four directions.

It is well known that there are three possible tessellations of a plane with regular polygons of the same kind: square, triangular, and hexagonal, corresponding to dividing a plane into regular squares, triangles, and hexagons, respectively. Based on this observation, some computer and communication networks has been built. The square tessellation is the basis for mesh-connected computers. The triangle tessellation is the basis to define hexagonal mesh multiprocessors [2], [15]. The hexagonal tessellation is the basis to define the honeycomb meshes [1], [11].

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Actually, Stojmenovic [11] introduced three different honeycomb meshes, namely honeycomb rectangular mesh, honeycomb rhombic mesh, and honeycomb hexagonal mesh. Most of theses meshes are not regular. To remedy these drawbacks, honeycomb rectangular torus, honeycomb rhombic torus, and honeycomb hexagonal torus are proposed [11]. Any such torus is 3-regular. Yet such torus is not hamiltonian unless it is small in size [8]. Moreover, all honeycomb tori are not planar. In this paper, we propose a variation of honeycomb meshes, called spider webs.

In the following section, we give some graph terms that are used in this paper and a formal definition of spider web networks. The spider graph $SW(m,n)$ is a bipartite graph with bipartition $C$ and $D$. Moreover, the honeycomb mesh $HREM(m,n)$ forms a spanning subgraph of $SW(m,n)$. In section 3, we prove that $SW(m,n)-e$ is hamiltonian for any $e \in E$. In section 4, we prove that $SW(m,n) - \{c,d\}$ remains hamiltonian for any $c \in C$ and $d \in D$. These hamiltonian properties are optimal. A conclusion is given in the final section.

## 2 Spider web networks

Usually, computer networks are represented by graphs where nodes represent processors and edges represent links between processors. In this paper, a network is represented as an undirected graph. For the graph definition and notation, we follow [3]. $G = (V,E)$ is a *graph* if $V$ is a finite set and $E$ is a subset of $\{(a,b) \mid (a,b)$ is an unordered pair of $V\}$. We say that $V$ is the *node set* and $E$ is the *edge set* of $G$. Two nodes $a$ and $b$ are *adjacent* if $(a,b) \in E$.

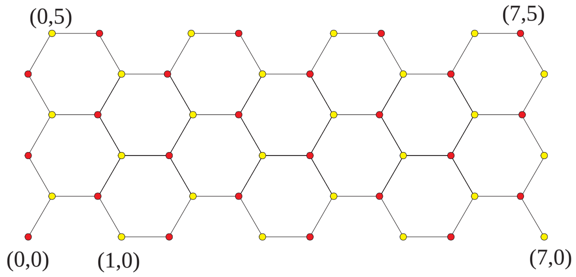


Figure 1: HREM(8,6).

The *honeycomb rectangular mesh* HREM$(m,n)$ is the graph with the node set $\{(i,j) \mid 0 \leq i < m, 0 \leq j < n\}$ such that $(i,j)$ and $(k,l)$ are adjacent if they satisfy one of the following conditions:

1. $i = k$ and $j = l \pm 1$;
2. $j = l$ and $k = i + 1$ if $i + j$ is odd; and
3. $j = l$ and $k = i - 1$ if $i + j$ is even.

For example, the honeycomb rectangular mesh $HREM(8,6)$ is shown in Figure 1.

A *spider web network* SW$(m,n)$ is the graph with the vertex set $\{(i,j) \mid 0 \leq i < m, 0 \leq j < n\}$ such that $(i,j)$ and $(k,l)$ are adjacent if they satisfy one of the following conditions:

1. $i = k$ and $j = l \pm 1$;
2. $j = l$ and $k = [i+1]_m$ if $i + j$ is odd or $j = n - 1$; and
3. $j = l$ and $k = [i-1]_m$ if $i + j$ is even or $j = 0$.

For example, the spider graph $SW(8,6)$ is shown in Figure 2(a). Another layout of $SW(8,6)$ is shown in Figure 2(b) with the dashed lines

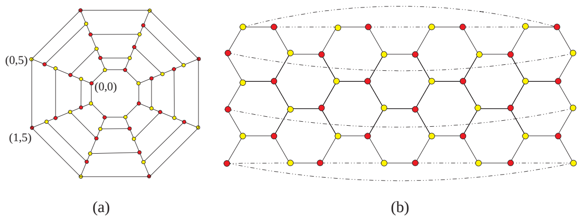


Figure 2: SW(8,6).

indicating those edges of $SW(m,n)$ that is not in $HREM(m,n)$. Obviously, $HREM(m,n)$ is a spanning subgraph of $SW(m,n)$. The *inner cycle* of $SW(m,n)$ is $\langle(0,0),(1,0),\ldots,(m-1,0),(0,0)\rangle$ whereas the *outer cycle* of $SW(m,n)$ is $\langle(0,n-1),(1,n-1),\ldots,(m-1,n-1),(0,n-1)\rangle$. It is obvious that any spider web network is a planar 3-regular bipartite graph. A vertex $(i,j)$ is labeled black when $i+j$ is even and white if otherwise.

The hamiltonian properties of a network are the major requirement in designing the topology of networks. For example, "token ring" approach is used in distributed operating systems. Fault tolerance is also desirable in massive parallel systems that have a relatively high probability of failure.

A *path* is a sequence of nodes such that two consecutive nodes are adjacent. A path is delimited by $\langle x_0, x_1, x_2, ..., x_{n-1}\rangle$. We use $P^{-1}$ to denote the path $\langle x_{n-1}, x_{n-2}, ..., x_1, x_0\rangle$ if $P$ is the path $\langle x_0, x_1, x_2, ..., x_{n-1}\rangle$. A path is called a *hamiltonian path* if its nodes are distinct and span $V$. A *cycle* is a path of at least three nodes such that the first node is the same as the last node. A cycle is called a *hamiltonian cycle* if its nodes are distinct except for the first node and the last node and if they span $V$. A *hamiltonian graph* is a graph with a hamiltonian cycle. The honeycomb rectangular mesh $HREM(8,6)$ is not hamiltonian because $\deg_{HREM(8,6)}(0,0) = 1$.

A graph $G = (V,E)$ is 1-*edge hamiltonian* if $G - e$ is hamiltonian for any $e \in E$. Obviously, any 1-edge hamiltonian graph is hamiltonian. A 1-edge hamiltonian graph $G$ is *optimal* if it contains the least number of edges among all 1-edge hamiltonian graphs with the same number of vertices as $G$. A graph $G = (V,E)$ is 1-*node hamiltonian* if $G - v$ is hamiltonian for any $v \in V$. A 1-node hamiltonian graph $G$ is *optimal* if it contains the least number of edges among all 1-node hamiltonian graphs with the same number of vertices as $G$. A graph $G = (V,E)$ is 1-*hamiltonian* if it is 1-edge hamiltonian and 1-node hamiltonian. A 1-hamiltonian graph $G$ is *optimal* if it contains the least number of edges among all 1-hamiltonian graphs with the same number of vertices as $G$. The study of optimal 1-hamiltonian graphs is motivated by the design of optimal fault-tolerant token rings in computer networks. A lot of optimal 1-hamiltonian graphs have been proposed [6, 10, 12]. Obviously, $deg_G(x) \geq 3$ for any vertex $x$ in a 1-edge hamiltonian, 1-node hamiltonian, or 1-hamiltonian graph $G$.

However, any bipartite graph is not 1-hamiltonian. Any cycle of a bipartite graph contains the same number of vertices in each partite set. Thus, the deletion of a vertex from a hamiltonian bipartite graph results in a non-hamiltonian graph. Let $G$ be a bipartite graph with bipartition $C$ and $D$. We use $\mathcal{F}(G)$ to denote $\{\{c,d\} \mid c \in C, d \in D\}$. A hamiltonian bipartite graph is $1_p$-*hamiltonian* if $G - F$ remains hamiltonian for any $F \in \mathcal{F}(G)$. Obviously, $deg_G(x) \geq 3$ for any vertex $x$ in a $1_p$-hamiltonian graph $G$. A $1_p$-hamiltonian graph $G$ is *optimal* if it contains the least number of edges among all $1_p$-hamiltonian graphs with the same number of vertices as $G$.

# 3 A recursive property of $SW(m, n)$

By the definition of the spider web network, $SW(m, n+2)$ can be constructed from $SW(m, n)$ as follows: Let $S$ denote the edge subset $\{((i, n-1), ([i-1]_m, n-1)) \mid i = 0, 2, 4, ..., m-2\}$ of $SW(m, n)$. Let $SW^*(m, n)$ denote the spanning subgraph of $SW(m, n)$ with edge set $E(SW(m,n)) - S$. Let $V^n = \{(i, k) \mid 0 \leq i < m; k = n, n+1\}$, and $E^n = \{((i, k), (i, k+1)) \mid 0 \leq i < m; k = n-1, n\} \bigcup \{((i, n), ([i-1]_m, n)) \mid i = 0, 2, 4, .., m-2\} \bigcup \{((i, n+1), ([i+1]_m, n+1)) \mid 0 \leq i < m\}$. Then $V(SW(m, n+2)) = V(SW(m, n)) \bigcup V^n$, $E(SW(m, n+2)) = (E(SW(m, n)) - S) \bigcup E^n$. For this reason, we can view $SW(m, n)$ as a substructure of $SW(m, n+2)$ if there is no confusion.

Suppose that $\mathcal{C}$ is a hamiltonian cycle of $SW(m, n) - F'$ for some faulty set $F' \subset V(SW^*(m, n)) \cup E(SW^*(m, n))$ such that $(i, n-1)$ is fault free for some $0 \leq i < m$. Now, we are going to construct a hamiltonian cycle of $SW(m, n+2)$ as follows:

**Case 1: there is some edge in $S \cap E(\mathcal{C})$.** Then we can pick an edge $((r, n-1), ([r-1]_m, n-1) \in \mathcal{C}$ for some even integer $0 \leq r < m-1$. For $0 \leq i \leq m-2$, we define $e^* = (([r+i]_m, n-1), ([r+i+1]_m, n-1))$, and $Q_i$ as

$$\begin{aligned}
&If\ [r+i]_2 = 0 &:& \quad Q_i = \langle([r+i]_m, n+1), ([r+i+1]_m, n+1)\rangle \\
&If\ [r+i]_2 = 1 &:& \quad Q_i = \langle([r+i]_m, n+1), ([r+i+1]_m, n+1)\rangle \\
&&& \qquad\qquad when\ e^* \in \mathcal{C} \\
&&& \quad Q_i = \langle([r+i]_m, n+1), ([r+i]_m, n), \\
&&& \qquad ([r+i+1]_m, n), ([r+i+1]_m, n+1)\rangle \\
&&& \qquad\qquad when\ e^* \notin \mathcal{C}.
\end{aligned}$$

Then set the path $Q$ as $\langle(r, n+1), Q_0, ([r+1]_m, n+1), Q_1, ([r+2]_m, n+1) \cdots ([r-2]_m, n+1), Q_{m-2}, ([r-1]_m, n+1)\rangle$.

Then, we perform the following algorithm on $\mathcal{C}$:

**Algorithm3 Extend**($\mathcal{C}$)

1. Replace those edges $((i, n-1), ([i-1]_m, n-1)) \in \mathcal{C}$, where $i \neq r$ and $i$ is even, with the path $\langle(i, n-1), (i, n), ([i-1]_m, n), ([i-1]_m, n-1)\rangle$.
2. Replace the edge $((r, n-1), (r-1, n-1))$ with the path $\langle(r, n-1), (r, n), (r, n+1), Q, (r-1, n+1), (r-1, n), (r-1, n-1)\rangle$.

Obviously, the resultant of **Algorithm 3** is a hamiltonian cycle of $SW(m, n+2) - F$.

**Case 2: there is no edge in $S \cap E(\mathcal{C})$.** Obviously, $((i, n-1), (i-1, n-1)) \in \mathcal{C}$ for every odd $i$ with $1 \leq i < m$. The hamiltonian cycle of $SW(m, n+2) - F$ can be easily constructed by replacing every $((i, n-1), (i-1, n-1))$, where $i$ is odd and $1 \leq i < m$, with the path $\langle(i, n-1), (i, n), (i, n+1), (i-1, n+1), (i-1, n), (i-1, n-1)\rangle$.

Thus, we have the following theorem.

**Theorem 3.1** *Assume that $F'$ is a faulty subset of $V(SW^*(m, n)) \cup E(SW^*(m, n))$ such that some*

$(i, n-1)$ *with* $0 \le i < m$ *is faulty free. Then* $SW(m, n+2) - F'$ *is hamiltonian if* $SW(m,n) - F'$ *is hamiltonian.*

## 4 $SW(m,n)$ is 1-edge hamiltonian

For $j = 0$ or $n-1$, $I_j(i,k)$ denotes $\langle (i,j), ([i+1]_m, j), ([i+2]_m, j), \ldots, (k,j) \rangle$, and $I_j^{-1}(i,k)$ denotes $\langle (k,j), ([k-1]_m, j), ([k-2]_m, j), \ldots, (i,j) \rangle$. In addition, let $H_i(j,k)$ denote the path $\langle (i,j), (i,j+1), (i,j+2), \ldots, (i,k) \rangle$, and $H_i^{-1}(j,k) = \langle (i,k), (i,k-1), \ldots, (i,j) \rangle$ for $0 \le i < m$, $0 \le j, k < n$.

**Theorem 4.1** *Any* $SW(m,n)$ *is 1-edge hamiltonian.*

**Proof.** We prove this theorem by induction. We first prove $SW(m,2)$ is 1-edge hamiltonian. Let $e$ be an edge of $SW(m,2)$. By the symmetric property of $SW(m,2)$, we may assume that $e$ is either $((0,0),(m-1,0))$ or $((i,0),(i,1))$ with $i \ne 0, m-1$. Obviously, $\langle (0,0), I_0(0,m-1), (m-1,0), (m-1,1), I_1^{-1}(0,m-1), (0,1), (0,0) \rangle$ forms a hamiltonian cycle of $SW(m,2) - e$.

Let $P_i = \langle (i+1,0), (i,0), H_i(0,n-1), (i,n-1), (i-1,n-1), H_{i-1}^{-1}(0,n-1), (i-1,0) \rangle$. Now, we consider the case $n = 4$ with $e = ((i,1),(i,2))$ for some $0 \le i < m$. By the symmetric property of $SW(m,4)$, we may assume that $i = 0$. Obviously, $\langle (0,0), (0,1), (1,1), H_1(1,3), (1,3), (0,3), (0,2), (m-1,2), (m-1,3), (m-2,3), H_{m-2}^{-1}(1,3), (m-2,1), (m-1,1), (m-1,0), (m-2,0), P_{m-3}, (m-4,0), P_{m-5}, (m-6,0), \ldots, P_3, (2,0), (1,0), (0,0) \rangle$ forms a hamiltonian cycle of $SW(m,4) - e$.

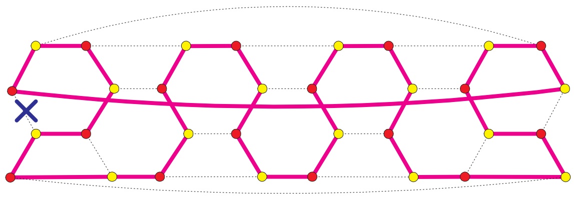

Figure 3: Illustration of Theorem 4.1.

By inductive assumption, suppose that the theorem is true for $n = k$ with $k \ge 2$, and $k \ge 4$ with $e = ((i,k-3),(i,k-2))$ for some $0 \le i < m$. Suppose that $e$ is an edge of $SW(m,k+2)$. Since the inner cycle and the outer cycle of $SW(m,k+2)$ are symmetric, we may assume that $e$ is in $SW^*(m,k)$. Then there exists a hamiltonian cycle of $SW(m,n) - e$. Applying Theorem 3.1, $SW(m,n+2) - e$ is hamiltonian.

Hence any spider web network $SW(m,n)$ is 1-edge hamiltonian. Figure 3 gives an illustration. □

## 5 $SW(m,n)$ is $1_p$-hamiltonian

**Lemma 5.1** $SW(m,2)$ *is* $1_p$*-hamiltonian for* $m \ge 4$.

**Proof.** Let $F \in \mathcal{F}(SW(m,2))$. By the symmetric property of $SW(m,2)$, we may assume that $(0,0) \in F$. So, the other vertex in $F$ is $(x,y)$, where $x+y$ is odd. Define two paths:

$$\begin{aligned} p_i(k,k+1) &= \langle (i-1,k), (i-1,k+1), (i,k+1), \\ &\quad (i,k), (i+1,k) \rangle \\ q_i(k+1,k) &= \langle (i-1,k+1), (i-1,k), (i,k), \\ &\quad (i,k+1), (i+1,k+1) \rangle \end{aligned}$$

To simplify the notation, $p_i = p_i(0,1)$ and $q_i = q_i(1,0)$.

Suppose that $y = 1$. Then we have a hamiltonian cycle of $SW(m, 2) - F$:

$$\langle (1,0),(2,0),p_3,(4,0),p_5,(6,0),\ldots,(x,0), (x+1,0),p_{x+2},(x+3,0),p_{x+4},\ldots,(m-1,0), (m-1,1),(0,1),(1,1),(1,0)\rangle.$$

Suppose that $y = 0$. There exists a hamiltonian cycle of $SW(m, 2) - F$:

$$\langle (0,1),(1,1),q_2,(3,1),q_4,(5,1),\ldots,(x,1), (x+1,1),q_{x+2},(x+3,1),\ldots,q_{m-3},(m-2,1), (m-2,0),(m-1,0),(m-1,1),(0,1)\rangle.$$

Hence $SW(m, 2)$ is $1_p$-hamiltonian. □

**Lemma 5.2** *There exist $\frac{m}{2} - 1$ disjoint paths, $P_1^n, P_2^n, \ldots, P_{\frac{m}{2}-1}^n$, that span $SW^*(m,n) - \{(0,0)\}$ such that $P_l^n$ joins $(2l, n-1)$ to $(2l+1, n-1)$ for $1 \le l < \frac{m}{2} - 1$, and $P_{\frac{m}{2}-1}^n$ joins $(0, n-1)$ to $(m-2, n-1)$.*

**Proof.** We prove this lemma by induction. For $n = 2$, we set $P_l^2$ as $\langle (2l,1),(2l+1,1)\rangle$ for $1 \le l < \frac{m}{2} - 1$, and set $P_{\frac{m}{2}-1}^2$ as $\langle (0,1),(1,1),(1,0),I_0(1,m-1),(m-1,0),(m-1,1),(m-2,1)$. Obviously, $P_l^2$'s satisfy the requirement of the lemma for $0 \le l \le \frac{m}{2} - 1$. Now assume that the lemma holds for $n = k$, where $k$ is even. Then, there exist $\frac{m}{2} - 1$ disjoint paths, $P_1^k, P_2^k, \ldots, P_{\frac{m}{2}-1}^k$, that span $SW^*(m,k) - \{(0,0)\}$ such that $P_l^k$ joins $(2l, k-1)$ to $(2l+1, k-1)$ for $1 \le l < \frac{m}{2} - 1$, and $P_{\frac{m}{2}-1}^k$ joins $(0, k-1)$ to $(m-2, k-1)$.

Now, we set $P_l^{k+2}$ as $\langle (2l,k+1),(2l+1,k+1)\rangle$ for $1 \le l < \frac{m}{2} - 1$. Define $f_i = \langle (i,k-1),(i,k),(i+1,k),(i+1,k-1),P_{(i+1)/2}^k,(i+2,k-1)\rangle$ and set $P_{\frac{m}{2}-1}^{k+2}$ as:

$$\langle (0,k+1),(1,k+1),(1,k),(2,k),(2,k-1), P_1^k,(3,k-1),f_3,(5,k-1),f_5,(7,k-1),\ldots, f_{m-5},(m-3,k-1),(m-3,k),(m-2,k), (m-2,k-2),(P_{\frac{m}{2}-1}^k)^{-1},(0,k-1),(0,k), (m-1,k),(m-1,k+1),(m-2,k+1)\rangle.$$

$P_l^{k+2}$, $1 \le l \le \frac{m}{2} - 1$, satisfies the requirement of lemma. Hence the lemma is proved. See Figure 4 (a) for an illustration. □

**Lemma 5.3** *Assume that $r$ is an even integer, $0 < r \le m - 2$. There exist $\frac{r}{2}$ disjoint paths, $Q_1^n, Q_2^n, \ldots, Q_{\frac{r}{2}}^n$, that span $SW^*(m,n) - \{(r,0)\}$, such that $Q_l^n$ joins $(2l, n-1)$ to $(2l+1, n-1)$ for $1 \le l \le \frac{r}{2} - 1$, and $Q_{\frac{r}{2}}^n$ joins $(0, n-1)$ to $(r, n-1)$.*

**Proof.** We prove this lemma by induction. For $n = 2$, we set $Q_l^2$ as $\langle (2l,1),(2l,0),(2l+1,0),(2l+1,1)\rangle$ for $1 \le l \le \frac{r}{2} - 1$, and set $Q_{\frac{r}{2}}^2$ as $\langle (0,1),(1,1),(1,0),(0,0),(m-1,0),(m-1,1),q_{m-2}^{-1},(m-3,1),q_{m-4}^{-1},(m-5,1),\ldots,q_{r+2}^{-1},(r+1,1),(r,1)\rangle$. Obviously, $Q_l^2$'s satisfy the requirement of the lemma for $1 \le l \le \frac{r}{2}$. We assume that the lemma holds for $n = k$ where $k$ is even. Then, there exist $\frac{r}{2}$ disjoint paths, $Q_l^k$, $1 \le l \le \frac{r}{2}$, that span $SW^*(m,k) - \{(r,0)\}$ such that $Q_l^k$ joins $(2l, k-1)$ to $(2l+1, k-1)$ for $1 \le l < \frac{r}{2}$, and $Q_{\frac{r}{2}}^k$ joins $(0, k-1)$ to $(r, k-1)$.

Now, we set $Q_l^{k+2}$ as $\langle (2l,k+1),(2l+1,k+1)\rangle$ for $1 \le l < \frac{r}{2}$. Define $g_i = \langle (i,k-1),Q_{i/2}^k,(i+1,k-1),(i+1,k),(i+2,k),(i+2,k-1)\rangle$, and set $Q_{\frac{r}{2}}^{k+2}$ as:

$$\langle (0,k+1),(1,k+1),(1,k),(2,k),(2,k-1), g_2,(4,k-1),g_4,(6,k-1),\ldots,g_{r-2},(r,k-1),$$

$(Q^k_{\frac{r}{2}})^{-1}, (0, k-1), (0, k), (m-1, k),$

$(m-1, k+1), q^{-1}_{m-2}(k+1, k), (m-3, k+1),$

$\ldots, q^{-1}_{r+2}(k+1, k), (r+1, k+1), (r, k+1)\rangle.$

Obviously, $Q^{k+2}_l$, for $1 \le l \le \frac{r}{2}$ satisfies the requirement of lemma. Hence the lemma is proved. See Figure 4 (b) for an illustration, where $r = 4$. □

**Lemma 5.4** *Assume that $s$ is a positive odd integer. There exist $\frac{m}{2} - 1$ disjoint paths, $R^n_l$, where $1 \le l < \frac{m}{2}$ that span $SW^*(m, n) - \{(s, 1)\}$ such that $R^n_l$ joins $(2(l-1), n-1)$ to $(2l-1, n-1)$ for $l \ne \frac{s+1}{2}$, and $R_{\frac{s+1}{2}}$ joins $(s-1, n-1)$ to $(m-2, n-1)$.*

**Proof.** We prove this lemma by induction. For $n = 2$, we set

$$\begin{aligned} R_l &= \langle(2(l-1), 1), (2(l-1), 0), (2l-1, 0), \\ &\quad (2l-1, 1)\rangle \quad for\ 1 \le l \le \frac{s-1}{2}, \\ R_l &= \langle(2(l-1), 1), (2l-1, 1)\rangle \\ &\quad for\ \frac{s+3}{2} \le l \le \frac{m}{2} - 1. \end{aligned}$$

Besides, $R^2_{\frac{s+1}{2}}$ as $\langle(s-1, 1), (s-1, 0), I_0(s-1, m-1), (m-1, 0), (m-1, 1), (m-2, 1)\rangle$. Obviously, $R^2_l$'s satisfy the requirement of the lemma for $1 \le l \le \frac{m}{2} - 1$. Now assume that the lemma holds for $n = k$ where $k$ is even. Then, there exist $\frac{m}{2} - 1$ disjoint paths, $R^k_l$'s, that span $SW^*(m, k) - \{(s, 1)\}$ such that $R^k_l$ joins $(2(l-1), k-1)$ to $(2l-1, k-1)$ for $1 \le l < \frac{m}{2}, l \ne \frac{s+1}{2}$, and $R^k_{\frac{s+1}{2}}$ joins $(s-1, k-1)$ to $(m-2, k-1)$.

Now, we set $R^{k+2}_l$ as $\langle(2(l-1), k+1), (2l-1, k+1)\rangle$ for $1 \le l < \frac{m}{2}, l \ne \frac{s+1}{2}$. Define $g_i = \langle(i, k-1), R^k_{i/2}, (i+1, k-1), (i+1, k), (i+2, k), (i+2, k-1)\rangle$, and set $R^{k+2}_{\frac{s+1}{2}}$ as:

$\langle(s-1, k+1), (s, k+1), (s, k), (s+1, k),$

$(s+1, k-1), g_{s+1}, (s+3, k-1), \ldots, g_{m-4},$

$(m-2, k-1), (R^k_{(s+1)/2})^{-1}, (s-1, k-1),$

$g^{-1}_{s-3}, (s-3, k-1), \ldots, g^{-1}_0, (0, k-1), (0, k),$

$(m-1, k), (m-1, k+1), (m-2, k+1)\rangle.$

Since $R^{k+2}_l$, for $1 \le l \le \frac{s}{2}$, satisfies the requirement of lemma, the lemma is proved. See Figure 4 (c), where $s = 3$. □

**Lemma 5.5** *There exist $\frac{m}{2} - 1$ disjoint paths, $S^n_l$, where $1 \le l < \frac{m}{2}$ that span $SW^*(m, n) - \{(0, 1)\}$ such that $S^n_l$ joins $(2l+2, n-1)$ to $(2l+3, n-1)$ for $1 \le l \le \frac{m}{2} - 2$ and $S^n_{\frac{m}{2}-1}$ joins $(1, n-1)$ to $(3, n-1)$.*

**Proof.** We prove this lemma by induction. For $n = 2$, we set $S^2_l = \langle(2l+2, 1), (2l+3, 1)\rangle$ for $1 \le l \le \frac{m}{2} - 2$, and $S^2_{\frac{m}{2}-1}$ as $\langle(1, 1), (1, 0), (0, 0), (m-1, 0), I^{-1}_0(2, m-1), (2, 0), (2, 1), (3, 1)\rangle$. Obviously, $S^2_l$'s satisfy the requirement of the lemma for $1 \le l \le \frac{m}{2} - 1$. Now assume that the lemma holds for $n = k$ where $k$ is even. Then, there exist $\frac{m}{2} - 1$ disjoint paths, $S^k_l$'s, that span $SW^*(m, k) - \{(0, 1)\}$ such that $S^k_l$ joins $(2l+2, k-1)$ to $(2l+3, k-1)$ for $1 \le l \le \frac{m}{2} - 2$, and $S^k_{\frac{m}{2}-1}$ joins $(1, k-1)$ to $(3, k-1)$.

Now, we set $S^{k+2}_l$ as $\langle(2l+2, k+1), (2l+3, k+1)\rangle$ for $1 \le l \le \frac{m}{2} - 2$. Define $h_i = \langle(i, k-1), (i, k), (i+1, k), (i+1, k-1), S^k_{\frac{i-1}{2}}, (i+2, k-1)\rangle$, and set $S^{k+2}_{\frac{m}{2}-1}$ as:

$\langle(1, k+1), (0, k+1), (0, k), (m-1, k),$

$(m-1, k-1), h^{-1}_{m-3}, (m-3, k-1), h^{-1}_{m-5},$

$(m-5, k-1), \ldots, h^{-1}_3, (3, k-1), (S^k_{\frac{m}{2}-1})^{-1},$

$(1, k-1), (1, k), (2, k), (2, k+1), (3, k+1)\rangle$.

$S_l^{k+2}$, $1 \le l \le \frac{m}{2} - 1$, satisfies the requirement of lemma, so the lemma is proved. See Figure 4 (d) for an illustration. □

**Lemma 5.6** *Assume that $t$ is an even integer, $0 < t \le m-2$. There exist $\frac{m}{2} - 1$ disjoint paths, $T_l^n$, where $1 \le l < \frac{m}{2}$ that span $SW^*(m, n) - \{(t, 1)\}$ such that $T_l^n$ joins $(2l, n-1)$ to $(2l+1, n-1)$ for $1 \le l \le \frac{m}{2} - 1$ and $l \neq \frac{t}{2}$, and $T_{\frac{t}{2}}^n$ joins $(1, n-1)$ to $(t+1, n-1)$.*

**Proof.** We prove this lemma by induction. For $n = 2$, we set $T_{\frac{t}{2}}^2 = \langle(1, 1), (0, 1), (0, 0), I_0(0, t+1), (t+1, 0), (t+1, 1)\rangle$.

$$T_l = \langle(2l, 1), (2l+1, 1)\rangle \quad for\ 1 \le l \le \frac{t-2}{2},$$
$$T_l = \langle(2l, 1), (2l, 0), (2l+1, 0), (2l+1, 1)\rangle \quad for\ \frac{t+2}{2} \le l \le \frac{m-2}{2}.$$

Obviously, $T_l^2$'s satisfy the requirement of the lemma for $1 \le l \le \frac{m}{2} - 1$. Now assume that the lemma holds for $n = k$ where $k$ is even. Then, there exist $\frac{m}{2} - 1$ disjoint paths, $T_l^k$'s, that span $SW^*(m, k) - \{(t, 1)\}$ such that $T_l^k$ joins $(2l, k-1)$ to $(2l+1, k-1)$ for $1 \le l \le \frac{m}{2} - 1$ and $l \neq \frac{t}{2}$, and $T_{\frac{t}{2}}^k$ joins $(1, k-1)$ to $(t+1, k-1)$.

Now, we set $T_l^{k+2}$ as $\langle(2l, k+1), (2l+1, k+1)\rangle$ for $1 \le l \le \frac{m}{2} - 1$, and $l \neq \frac{t}{2}$. Define $h_i = \langle(i, k-1), (i, k), (i+1, k), (i+1, k-1), T_{\frac{i+1}{2}}^k, (i+2, k-1)\rangle$, and set $T_{\frac{t}{2}}^{k+2}$ as:

$$\langle(1, k+1), (0, k+1), (0, k), (m-1, k), (m-1, k-1), h_{m-3}^{-1}, (m-3, k-1), h_{m-5}^{-1}, (m-5, k-1), \dots, h_{t+1}^{-1}, (t+1, k-1), (T_{\frac{t}{2}}^k)^{-1}, (1, k-1), h_1, (3, k-1), \dots, h_{t-3}, (t-1, k-1), (t-1, k), (t, k), (t, k+1), (t+1, k+1)\rangle.$$

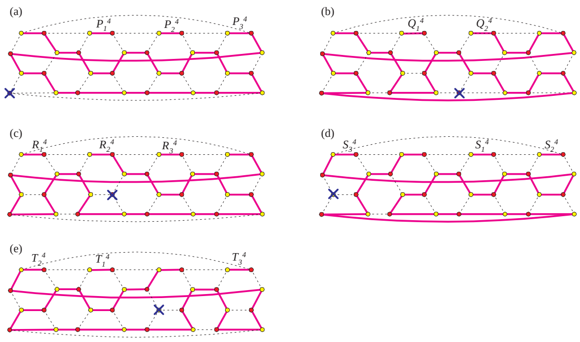

Figure 4: An illustration for Lemma 5.2 to Lemma 5.6

$T_l^{k+2}$, $1 \le l \le \frac{m}{2} - 1$, satisfies the requirement of lemma, so the lemma is proved. See Figure 4 (e), where $t = 4$. □

**Theorem 5.1** *$SW(m, n)$ is $1_p$-hamiltonian for any even integer with $m \ge 4, n \ge 2$.*

**Proof.** This theorem is proved by induction. By Lemma 5.1, $SW(m, 2)$ is $1_p$-hamiltonian. Assume that $SW(m, k)$ is $1_p$-hamiltonian for some positive integer $k$ with $k \ge 2$.

Now, we want to prove $SW(m, n+2)$ is $1_p$-hamiltonian. Let $F \in \mathcal{F}(SW(m, n+2))$. Obviously, one of the following cases holds: (1) $\{(i, j) \mid 0 \le i < m, j = n, n+1\} \cap F = \emptyset$, (2) $\{(i, j) \mid 0 \le i < m, j = 0, 1\} \cap F = \emptyset$, and (3) $|\{(i, j) \mid 0 \le i < m, j = n, n+1\} \cap F| = 1$ and $|\{(i, j) \mid 0 \le i < m, j = 0, 1\} \cap F| = 1$.

**Case 1:** $\{(i, j) \mid 0 \le i < m, j = n, n+1\} \cap F = \emptyset$. Then $F \in \mathcal{F}(SW(m, n))$. By induction, $SW(m, n) - F$ is hamiltonian. Applying Theorem 3.1, $SW(m, n+2) - F$ is hamiltonian.

**Case 2:** $\{(i,j) \mid 0 \le i < m, j = 0,1\} \cap F = \emptyset$. Since the inner cycle and the outer cycle are symmetric in any spider web network, $SW(m,n+2) - F$ is hamiltonian as case 1.

**Case 3:** $|\{(i,j) \mid 0 \le i < m, j = n, n+1\} \cap F| = 1$ and $|\{(i,j) \mid 0 \le i < m, j = 0,1\} \cap F| = 1$. By the symmetric property of the spider web networks, we have the following five cases: (3.1) $F = \{(0,0),(0,n+1)\}$, (3.2) $F = \{(r,0),(0,n+1)\}$ with $r$ an nonzero even integer, (3.3) $F = \{(s,1),(0,n+1)\}$ with $s$ an odd integer, (3.4) $F = \{(0,1),(0,n)\}$, and (3.5) $F = \{(t,1),(0,n)\}$ with $t$ an nonzero even integer.

**Case (3.1):** $F = \{(0,0),(0,n+1)\}$. By Lemma 5.2, there exist $\frac{m}{2} - 1$ disjoint paths, $P_1^n, P_2^n, \ldots, P_{\frac{m}{2}-1}^n$, that span $SW^*(m,n) - \{(0,0)\}$ such that $P_l^n$ joins $(2l, n-1)$ to $(2l+1, n-1)$ for $1 \le l < \frac{m}{2} - 1$, and $P_{\frac{m}{2}-1}^n$ joins $(0, n-1)$ to $(m-2, n-1)$.

Define $C_1(i) = \langle (i,n-1),(i,n),(i-1,n),(i-1,n-1),(P_{\frac{i-2}{2}}^n)^{-1},(i-2,n-1)\rangle$. Obviously, $\langle (0,n-1), P_{\frac{m}{2}-1}^n, (m-2,n-1), C_1(m-2), (m-4,n-1), \ldots, C_1(4), (2,n-1),(2,n),(1,n),(1,n+1), I_{n+1}(1,m-1), (m-1,n+1),(m-1,n),(0,n),(0,n-1)\rangle$ forms a hamiltonian cycle of $SW(m,n+2) - F$. See Figure 5 (a)

**Case (3.2):** $F = \{(r,0),(0,n+1)\}$. By Lemma 5.3, there exist $\frac{r}{2}$ disjoint paths, $Q_1^n, Q_2^n, \ldots, Q_{\frac{r}{2}}^n$, that span $SW^*(m,n) - \{(r,0)\}$ such that $Q_l^n$ joins $(2l, n-1)$ to $(2l+1, n-1)$ for $1 \le l < \frac{r}{2}$, and $Q_{\frac{r}{2}}^n$ joins $(0,n-1)$ to $(r,n-1)$.

Define $C_2(i) = \langle (i,n-1),(i,n),(i-1,n),(i-1,n-1),(Q_{\frac{i-2}{2}}^n)^{-1},(i-2,n-1)\rangle$, $B(i) \equiv \langle (i,n+1),(i,n),(i+1,n),(i+1,n+1),(i+2,n+1)\rangle$. Obviously, $\langle (0,n-1), Q_{\frac{r}{2}}^n, (r,n-1), C_2(r), (r-2,n-1), \ldots, C_2(4), (2,n-1),(2,n),(1,n),(1,n+1), I_{n+1}(1,r+1), (r+1,n+1), B(r+1), (r+3,n+1), \ldots, B(m-3), (m-1,n+1),(m-1,n),(0,n),(0,n-1)\rangle$ forms a hamiltonian cycle of $SW(m,n+2) - F$. See Figure 5 (b), where $r = 4$.

**Case (3.3):** $F = \{(s,1),(0,n+1)\}$. By Lemma 5.4, there exist $\frac{m}{2} - 1$ disjoint paths, $R_1^n, R_2^n, \ldots, R_{\frac{m}{2}-1}^n$, that span $SW^*(m,n) - \{(s,1)\}$ such that $R_l^n$ joins $(2l-2, n-1)$ to $(2l-1, n-1)$ for $1 \le l < \frac{m}{2}$ and $l \ne \frac{s+1}{2}$, and $R_{\frac{s+1}{2}}^n$ joins $(s-1,n-1)$ to $(m-2,n-1)$.

Define $C_3(i) = \langle (i,n-1),(i,n),(i-1,n),(i-1,n-1),(R_{\frac{i}{2}}^n)^{-1},(i-2,n-1)\rangle$, $C_3'(i) \equiv \langle (i,n-1), R_{\frac{i+2}{2}}^n, (i+1,n-1),(i+1,n),(i+1,n+1),(i+2,n+1),(i+2,n),(i+2,n-1)\rangle$. Obviously, $\langle (s-1,n-1), R_{\frac{s+1}{2}}^n, (m-2,n-1), C_3(m-2), (m-4,n-1), C_3(m-4), (m-6,n-1) \ldots C_3(s+3), (s+1,n-1),(s+1,n),(s,n),(s,n+1), I_{n+1}(s,m-1), (m-1,n+1),(m-1,n),(0,n),(0,n-1), C_3'(0), (2,n-1), C_3'(2), (4,n-1) \ldots, C_3'(s-3), (s-1,n-1)\rangle$ forms a hamiltonian cycle of $SW(m,n+2) - F$. See Figure 5 (c), where $s = 3$.

**Case (3.4):** $F = \{(0,1),(0,n)\}$. By Lemma 5.5, there exist $\frac{m}{2} - 1$ disjoint paths, $S_1^n, S_2^n, \ldots, S_{\frac{m}{2}-1}^n$, that span $SW^*(m,n) - \{(0,1)\}$ such that $S_l^n$ joins $(2l+2, n-1)$ to $(2l+3, n-1)$ for $1 \le l \le \frac{m}{2} - 2$, and $S_{\frac{m}{2}-1}^n$ joins $(1,n-1)$ to $(3,n-1)$.

Define $C_4(i) = \langle (i,n-1),(i,n),(i,n+1),(i+1,n+1),(i+1,n),(i+1,n-1), S_{\frac{i-1}{2}}^n, (i+2,n-1)\rangle$. Obviously, $\langle (1,n-1), S_{\frac{m}{2}-1}^n, (3,n-1), C_4(3), (5,n-1), C_4(5), (7,n-1), \ldots, C_4(m-3), (m-1,n-1),(m-1,n),(m-1,n+1),(0,n+$

$1), (1, n+1), (2, n+1), (2, n), (1, n), (1, n-1)\rangle$ forms a hamiltonian cycle of $SW(m, n+2) - F$. See Figure 5 (d).

**Case (3.5):** $F = \{(t, 1), (0, n)\}$. By Lemma 5.6, there exist $\frac{m}{2} - 1$ disjoint paths, $T_1^n, T_2^n, \ldots, T_{\frac{m}{2}-1}^n$, that span $SW^*(m, n) - \{(t, 1)\}$ such that $T_l^n$ joins $(2l, n-1)$ to $(2l+1, n-1)$ for $1 \le l \le \frac{m}{2} - 1$ and $l \ne \frac{t}{2}$, and $T_{\frac{t}{2}}^n$ joins $(1, n-1)$ to $(t+1, n-1)$.

Define $C_5(i) = \langle (i, n-1), (i, n), (i, n+1), (i+1, n+1), (i+1, n), (i+1, n-1), T_{\frac{i+1}{2}}^n, (i+2, n-1)\rangle$, and $C_5'(i) = \langle (i, n-1), (T_{\frac{i-1}{2}}^n)^{-1}, (i-1, n-1), (i-1, n), (i-2, n), (i-2, n-1)\rangle$. Obviously, $\langle (1, n-1), T_{\frac{t}{2}}, (t+1, n-1), C_5(t+1), (t+3, n-1), C_5(t+3), (t+5, n-1), \ldots, C_5(m-3), (m-1, n-1), (m-1, n), (m-1, n+1), (0, n+1), I_{n+1}(0, t), (t, n+1), (t, n), (t-1, n), (t-1, n-1), C_5'(t-1), (t-3, n-1), C_5'(t-3), (t-5, n-1), \ldots, C_5'(3), (1, n-1)\rangle$ forms a hamiltonian cycle of $SW(m, n+2) - F$. See Figure 5 (e), where $t = 4$

Thus we have proved the theorem. □.

# 6 Concluding remarks

Since
the honeycomb rectangular mesh $HREM(m, n)$ is a spanning subgraph of $SW(m, n)$, the spider web network can be viewed as a variation of the honeycomb meshes. The spider web network we proposed are 3-regular planar graphs. Moreover, they are 1-edge hamiltonian and $1_p$-hamiltonian. Since the spider web network is 3-regular, it is optimal.

It is very easy to see that the diameter of the

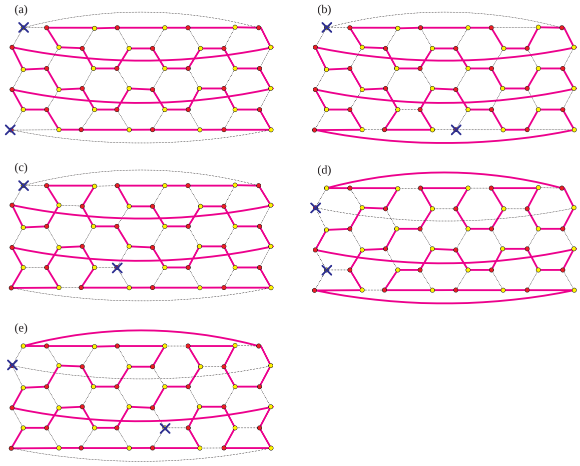


Figure 5: Illustration for Theorem 5.1, case (3.1)-(3.5).

spider web network $SW(m, n)$ is $O(m + n)$. By choosing $m = O(n)$, the diameter of $SW(m, n)$ is $O(\sqrt{N})$ where $N = mn$ is the number of vertices in $SW(m, n)$. It would be interesting to find other planar, 3-regular, 1-edge hamiltonian, and $1_p$-hamiltonian graphs with smaller diameters.

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