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On the panconnected properties of the Augmented cubes

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***Abstract*— Many topologies have been proposed to balance the performance and some cost parameters. Hypercubes are widely studied in interconnection networks [8], [9]. Augmented cubes are derivatives of hypercubes with good geometric nature and retain all the favorable properties of the hypercube. In this paper, we consider the path embedding problem with fixed endpoints in the Augmented cube AQ_n. For any two distinct vertices x, y in AQ_n, let $d_{AQ_n}(x, y)$ be the length of the shortest path between x and y. We show that there exists a path of length l joining x and y for every l satisfying $d_{AQ_n}(x, y) \leq l \leq |V(AQ_n)| - 1$. As a consequence of it, AQ_n is edge pancyclic.**

Keywords: panconnected, panconnectivity, Augmented cube, path embedding, ring embedding, pancyclic.

1. Introduction

For the graph definition and notation we follow [1]. $G = (V, E)$ is a graph if V is a finite set and E is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. Two vertices u and v are *adjacent* if $(u, v) \in E$. A path is a sequence of adjacent vertices, written as $\langle v_0, v_1, v_2, \ldots, v_m\rangle$, in which all the vertices $v_0, v_1, \ldots, v_m$ are distinct except possibly $v_0 = v_m$. We also write the path $\langle v_0, P, v_m\rangle$, where $P = \langle v_0, v_1 \ldots, v_m\rangle$. The *length* of a path P, denoted by $len(P)$, is the number of edges in P. Let u and v be two vertices of G. The *distance* between u and v, denoted by $d_G(u, v)$, is the length of the shortest path of G joining u and v.

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A graph G is *panconnected* if each pair of distinct vertices u, v are joined by a path of length l, $d_G(u, v) \leq l \leq |V(G)| - 1$. Broersma [2], Kanetkar [7], and Seng [10] et al. studied this problem on some connected graphs. In this paper, we consider the path embedding problem with fixed endpoints in the Augmented cube AQ_n. We show that AQ_n is panconnected for $n \geq 2$.

A *cycle* is a path with at least three vertices such that the first vertex is the same as the last one. The girth of G, $g(G)$, is the length of the shortest cycle in G. The ring embedding problem, which deals with all the possible lengths of the cycles, is investigated in a lot of interconnection networks [4], [5], [6]. A graph is *pancyclic* if it contains a cycle of every length from 3 to $|V(G)|$ inclusive. By definition, the girth of any pancyclic graph is 3. Hence, a graph with large girth is not pancyclic. Furthermore, a graph is called *edge-pancyclic* if every edge lies on a cycle of length l for all $l = 3, 4, \ldots, |V(G)|$.

2. Definitions and Notations

The following is the recursive definition of the n-dimensional Augmented cube AQ_n.

Definition 1: [3] Let $n \geq 1$ be an integer. The augmented cube AQ_n of dimension n has 2^n vertices, each labelled by an n-bit binary string $a_1a_2 \ldots a_n$. We define $AQ_1 = K_2$. For $n \geq 2$, AQ_n is obtained by taking two copies of the augmented cube AQ_{n-1}, denoted by AQ^0_{n-1} and AQ^1_{n-1}, and adding $2 \times 2^{n-1}$ edges between the two as follows:

Let $V(AQ^0_{n-1}) = \{0a_{n-2}a_{n-3} \ldots a_0 : a_i = 0$ or $1, 0 \leq i \leq n-2\}$ and $V(AQ^1_{n-1}) = \{1b_{n-2}b_{n-3} \ldots b_0 : b_i = 0$ or $1, 0 \leq i \leq n-2\}$. A vertex $a = 0a_{n-2}a_{n-3} \ldots a_0$ of AQ^0_{n-1} is joined to a vertex $b = 1b_{n-2}b_{n-3} \ldots b_0$ of AQ^1_{n-1} iff for every $i, 0 \leq i \leq n-2$, either

(1) $a_i = b_i$; in this case, (a, b) is called a hypercube edge, or

(2) $a_i = \overline{b}_i$; in this case, (a, b) is called a complete edge.

The augmented cubes AQ_1, AQ_2, and AQ_3 are shown as Fig. 1.

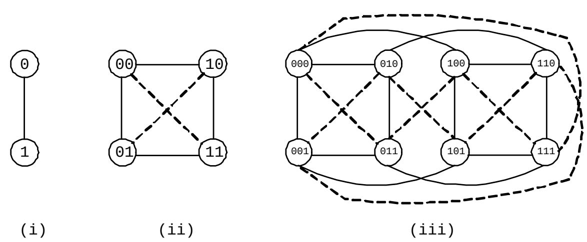

Fig. 1. Augmented cubes AQ_1, AQ_2, and AQ_3.

We write this recursive construction of AQ_n symbolically as $AQ_n = AQ^0_{n-1} \bigotimes AQ^1_{n-1}$. Let $u = u_{n-1}u_{n-2} \ldots u_1u_0$ be an n-bits binary string. For $0 \leq k \leq n-1$, we use u^h_k to denote the binary string $v_{n-1}v_{n-2} \ldots v_1v_0$ such that $v_k = \overline{u}_k$ and $u_i = v_i$ for all $i \neq k$. For $1 \leq k \leq n-1$, we use u^c_k to denote the binary string $w_{n-1}w_{n-2} \ldots w_1w_0$ such that $w_j = \overline{u}_j$ for $j \leq k$ and $w_i = v_i$ for $i \geq k+1$. Moreover, u^h_k and u^c_k are both k-dimensional neighbors of u. Also, u^h (u^c) is used to represent $(n-1)$-dimensional neighbor u^h_{n-1} (respectively u^c_{n-1}) of u. And, (u, v) is called a edge of dimension i if $v = u^h_i$ or $v = u^c_i$.

Let u, v be two distinct vertices of AQ_n. There are some known properties for the shortest paths joining them as follows.

Lemma 1: [3] Let (u, v) be an i-dimensional edge in $E(AQ_n)$, $0 \leq i \leq n-1$. Assume k be an integer such that $i \neq k$. Then

(1) $(u^h_k, v^h_k) \in E(AQ_n)$ for $0 \leq k \leq n-1$, and

(2) $(u^c_k, v^c_k) \in E(AQ_n)$ for $1 \leq k \leq n-1$.

Lemma 2: [3] Let $u, v \in AQ_n$.

(1) If $u, v \in AQ^0_{n-1}$ (AQ^1_{n-1}), then there exists a shortest path joining u and v in AQ_n with all its vertices in AQ^0_{n-1} (respectively, AQ^1_{n-1}).

(2) Let $u \in AQ^0_{n-1}$ and $v \in AQ^1_{n-1}$. Then,

(i) There exists a shortest path S joining u and v in AQ_n with all its vertices (except u) in AQ^1_{n-1}.

(ii) There exists a shortest path S joining u and v in AQ_n with all its vertices (except v) in AQ^0_{n-1}.

3. The Panconnected Property of AQ_n

For AQ_2, any two distinct vertices x, y are adjacent and there exists a path of length l between them for each l, $1 \leq l \leq |V(AQ_2)| - 1 = 3$. That is, AQ_2 is panconnected. Then, we show that AQ_n is panconnected for $n \geq 2$.

Theorem 1: For $n \geq 2$, AQ_n is panconnected.

Proof: We prove this theorem by induction on n. Clearly, this lemma holds for $n = 2$. Assume that it holds for some $n - 1 \geq 2$. We now show that it holds for n. For n-dimensional cube AQ_n, it is obtained from two $(n-1)$-dimensional cubes AQ_{n-1}'s, AQ^0_{n-1} and AQ^1_{n-1}. Let $u = u_{n-1}u_{n-2} \ldots u_1u_0$ and $v = v_{n-1}v_{n-2} \ldots v_1v_0$ be any two vertices in AQ_n. Without loss of generality, let $u \in V(AQ^0_{n-1})$. We study two main cases: (1) $v \in V(AQ^0_{n-1})$ and (2) $v \in V(AQ^1_{n-1})$.

Case 1: $v \in V(AQ^0_{n-1})$.

By Lemma 2, there exists a shortest path joining u and v in AQ_n with all its vertices in AQ^0_{n-1}. By the induction hypothesis, there exists a path joining u and v in AQ^0_{n-1} for each l, $d_{AQ^0_{n-1}}(u, v) \leq l \leq 2^{n-1} - 1$.

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Suppose that $2^{n-1} \leq l \leq 2^n - 1$. Let P_0 be one of the longest paths of AQ^0_{n-1} joining u and v, and let $l_0 = len(P_0)$. Then $l_0 = 2^{n-1} - 1$. Let $l_1 = l - l_0 - 1$. Let (x, y) be any edge on P_0. We can write P_0 as $\langle u, P_{01}, x, y, P_{02}, v\rangle$. By definition, x^h and y^h are vertices in AQ^1_{n-1}; and, by Lemma 1, $d_{AQ^1_{n-1}}(x^h, y^h) = 1$. By induction hypothesis, there exists a path P_1 of length l_1 in AQ^1_{n-1} joining x^h and y^h. Thus, $\langle u, P_{01}, x, x^h, P_1, y^h, y, P_{02}, v\rangle$ is a path of length l in AQ_n joining u and v.

Case 2: $v \in V(AQ^1_{n-1})$

Here, we divide this case into two subcases according to whether $(u, v) \in E(AQ_n)$.

Subcase 2.1: $v \in V(AQ^1_{n-1})$ and $(u, v) \notin E(AQ_n)$.

Suppose that $d_{AQ_n}(x, y) \leq l \leq 2^{n-1}$. By Lemma 2, there exists a shortest path S joining u and v in AQ_n with all its vertices (except v) in AQ^0_{n-1}. Let v' be the neighbor vertex of v on S. That is, $v' \in V(AQ^0_{n-1})$. By the induction hypothesis, there exists a path P_0 of length $l - 1$ in AQ^0_{n-1} joining u and v'. Then $\langle u, P_0, v', v\rangle$ is a path of length l in AQ_n joining u and v.

Suppose that $2^{n-1}+1 \leq l \leq 2^n-1$. Let y be a neighbor of v in AQ^1_{n-1} and x be a neighbor of y in AQ^0_{n-1} with $x \neq u$. Then $d_{AQ_n}(x, y) = d_{AQ^1_{n-1}}(y, v) = 1$. Let P_0 be one of the longest paths of AQ^0_{n-1} joining u and x; and let $l_0 = len(P_0)$. Then $l_0 = 2^{n-1} - 1$. Let $l_1 = l - l_0 - 1$. By induction hypothesis, there exists a path P_1 of length l_1 in AQ^1_{n-1} joining y and v. Thus, $\langle u, P_0, x, y, P_1, v\rangle$ is a path of length l in AQ_n joining u and v.

Subcase 2.2: $v \in V(AQ^1_{n-1})$ and $(u, v) \in E(AQ_n)$.

That is, $d_{AQ_n}(u, v) = 1$. Without loss of generality, let $u = 0^n$. Then $v = u^h = 10^{n-1}$ or $v = u^c = 1^n$. Let $x = u^c$ (u^h) if $v = u^h$ (respectively $v = u^c$). It is not difficult to verify that $d(x, v) = 1$. Using the similar technique in case 2.1, all paths of length l, $2 \leq l \leq 2^n - 1$, joining u and v can be found in AQ_n.

Hence, the theorem follows. □

By Theorem 1, we have the following result.

Corollary 1: For $n \geq 2$, AQ_n is edge-pancyclic.

4. Conclusions

Augmented cubes have other good properties that have been demonstrated like vertex symmetry, maximum connectivity ($2n - 1$, equals to degree), best possible wide diameter ($\lceil \frac{n}{2} \rceil$+1), routing and broadcasting procedures with liner time complexity. The main result of this paper is embedding of paths of all lengths into the Augmented cubes. By the result, we observe that AQ_n, is edge pancyclic for $n \geq 2$. The question of embedding other important networks, like the Twisted cubes and Mӧbius cubes still remains open.

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