An Optimal Restricted Multi-Headed Disk System[⊕]

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Abstract

Disk system is important to operating system and database management system. Now, the multi-headed disk system is commercially available. There are many such disk systems used in our computers to improve the system performance. In this paper, we consider a restricted multi-headed disk system. The disk heads in this system are locked in a fixed number of cylinders, and they are distributed on the disk surface equally. We call these heads *restricted* since these disk heads must remain within the disk surface. According to our analysis, this disk system is optimal based upon the amortized complexity. Moreover, we obtain that this disk system performs more than k times as well as a system with only one head. Here k is the number of the disk heads used in our multi-headed disk system. This result matches the conclusions of the previous studies in which multi-headed disk systems have been analyzed based on some probability models.

Keywords: amortized analysis, on-line problem, disk system, multi-headed disk

1. Introduction

Disk system is important to operating system and database management system. Many studies of disk systems have been proposed. Most of them have only one head per moving arm in their disk systems. In other words, these disk systems are one-headed. Now, multi-headed disk systems are commercially available. These systems aim to minimize the expensive movement of the disk heads and consequently reduce the seek time. The disk heads in some of these systems are locked in a fixed number of cylinders and are must stayed within the disk surface. We call these disk heads *restricted heads* and the disk system *restricted multi-headed disk system*.

There have been many papers concerning with the restricted multi-headed disk system, especially the restricted two-headed disk system. In [5], a restricted two-headed disk system was examined using a simulation model. The model was driven by a random request sequence and a Shortest-Seek-Time-First (SSTF) scheduling policy. The analytical work in [1] assumed that the requests were served on the Fist-Come-First-Serviced (FCFS). The disk scheduling policy SCAN was used for file access in [4], and it applied combinatorial analysis to derive exact formulas for the expected head movement. In addition, we analyzed this disk system by amortized analysis in [3].

According to their analyses, the optimum head separation was found to be approximately one-half of the total number of cylinders. Moreover, in [1,3,4], a comparison with a one-headed disk system showed that if the two heads are optimally spaced, the mean seek distance is less than one-half of the value obtained with one head.

In this paper, we generalize our previous result in [3] and propose a restricted multi-headed disk system. We called it *PLAIN*. Its performance is analyzed by *amortized analysis* technique. The amortized analysis was proposed by Tarjan in 1985 [6]. It is a very useful tool for analyzing the time-complexity of performing a sequence of operations. Amortized analysis is suitable to analyze the performance of a disk system since it involves a sequence of requests. According to our analysis, *PLAIN* is an optimal restricted multi-headed disk system. Its amortized complexity is moreover equal to k times the optimal complexity obtained with one-headed disk system. Here k means the number of the disk heads used in *PLAIN*.

In the rest of this paper, Section 2 presents the restricted multi-headed disk system *PLAIN*. Its amortized analysis is analyzed in Section 3. A lower bound of the amortized complexity is derived in Section 4. Concluding remarks are given in Section 5.

2. Our Restricted Multi-Headed Disk System *PLAIN*

PLAIN is a restricted multi-headed disk system. Data in this system are stored on various cylinders. Suppose that there are k disk heads used in this system. Q is the total number of the cylinders. These cylinders are numbered from 0 to $Q-1$. The head separation in *PLAIN* is then Q/k. Since all k disk heads must be staid within 0, 1, ..., and $Q-1$ cylinders, the first head should be located between 0, 1, ..., and $Q/k -1$ cylinders, the second head should be located between Q/k, $Q/k +1$, ..., $2 \cdot Q/k-1$, and so on. Here we assume that Q can be divided by k and, initially, the k disk heads are respectively located on cylinder 0, Q/k, ..., and $(k-1) \cdot Q/k-1$.

At any time, there are a set of requests to retrieve data on disk system. This set of requests is called a *waiting queue* and these requests are called *waiting requests*. A disk scheduler selects one of waiting requests as the next request to be served. In *PLAIN*, the requests are served by scheduling policy SCAN. As shown in [2], SCAN always chooses the nearest request in the *sweep-direction*. Assuming that initially the sweep direction is outward, SCAN will not change this direction until the heads reach the outermost cylinders or until there is no waiting request in this direction, and vice versa.

For example, there is a five-headed disk system which total number of cylinders is 100, i.e., $k = 5$ and $Q = 100$. The five disk heads are then located on cylinders (0, 20, 40, 60, 80) initially. That is, at the beginning, the disk heads are located on cylinders 0, 20, 40, 60, and 80. Consider a sequence of requests to access data stored on cylinders 65, 15, 30, 17, 62, 75, 72, 30, 5, 80, 57, ... respectively. Suppose the length of the waiting queue is just equal to 4 at any time. For processing this sequence of requests, *PLAIN* first moves the disk heads from cylinders (0, 20, 40, 60, 80) to (5, 25, 45, 65, 85), then to (10, 30, 50, 70, 90), (15, 35, 55, 75, 95), (17, 37, 57, 77, 97), (12, 32, 52, 72, 92), (10, 30, 50, 70, 90), and so on. The total seek distance of this sequence of requests is then |0-5|+ |5-10|+ |10-15|+ |15-17|+ |17-12|+ |12-10|= 5 + 5+ + 5+ 2 + 5 +2 = 24. This schedule can be diagrammed in the following figure.

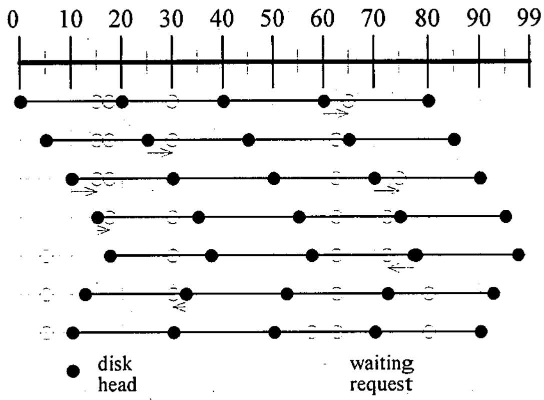

Figure 1: An Example of *PLAIN*

3. Amortized Analysis for *PLAIN*

Consider a sequence of m requests processed by a restricted k-headed disk system. During the entire process, these requests may keep coming in. If the waiting queue is longer than m, we will ignore those requests outside of the m requests. In other words, the maximum length of the waiting queue considered here is m. On the other hand, we assume that the minimum length of the waiting queue is W where $W \geq 2$. A disk scheduling policy would select a request in the waiting queue to process. Denote the i-th seek distance by t_i. Then the *amortized complexity* of a k-headed disk system is the worst case of $\sum_{i=1}^{m} t_i / m$.

The "*potential function*" technique [6] is useful in amortized analysis, and is employed in this paper. Consider a k-headed disk system. Let Φ_{i-1} and Φ_i be the potentials before and after the i-th serving, respectively. The amortized time a_i of this is defined as

$$a_i = t_i + \Phi_i - \Phi_{i-1}, \quad (1)$$

where t_i is the seek distance of the i-th served request. Summing the amortized time of all the m requests, we have

$$\sum_{i=1}^{m} a_i = \sum_{i=1}^{m} (t_i + \Phi_i - \Phi_{i-1})$$
$$= \sum_{i=1}^{m} t_i + \Phi_m - \Phi_0.$$

By deriving an upper bound A of a_i, we obtain an upper bound of $\sum_{i=1}^{m} t_i$ as follows:

$$\sum_{i=1}^{m} t_i = \sum_{i=1}^{m} a_i + \Phi_0 - \Phi_m$$
$$\leq m \cdot A + \Phi_0 - \Phi_m.$$

Then $m \cdot A + \Phi_0 - \Phi_m$ is shown to be an upper bound of $\sum_{i=1}^{m} t_i$. Averaging this result by m, the amortized complexity is then obtained.

Based on this technique, we define the potential function of *PLAIN* first. Consider the status after the i-th transaction, where $0 \leq i \leq m$. If the sweep-direction is not changed, N_i is defined as the number of requests having been served, and D_i is defined as the distance which the disk head has been moved in the current sweep (including this transaction); otherwise, N_i and D_i are set to zero. N_0 and D_0 are zero intuitively. The potential function of *PLAIN* is then defined as

$$\Phi_i = N_i \cdot \frac{1}{W}(\frac{Q}{k} - 1) - D_i.$$

Theorem 1: (Amortized Complexity of *PLAIN*)

The amortized complexity of PLAIN is no greater than $\frac{m+W-1}{m \cdot W}(\frac{Q}{k} - 1)$.

Proof:

Consider the i-th serving, where $0 \leq i \leq m$. Suppose that the seek distance to serve this request is t_i, the amortized time is a_i, and an upper bound of a_i is A. To derive A, the following two cases are considered:

[Case 1]: The sweep-direction is not changed after serving this request.

In this case, $N_i = N_{i-1} + 1$ and $D_i = D_{i-1} + t_i$. Then

$$a_i = t_i + \Phi_i - \Phi_{i-1}$$
$$= t_i + [\, N_i \cdot \frac{1}{W}(\frac{Q}{k} - 1) - D_i \,] - [\, N_{i-1} \cdot \frac{1}{W}(\frac{Q}{k} - 1) - D_{i-1} \,]$$
$$= t_i + [\, (N_{i-1}+1) \cdot \frac{1}{W}(\frac{Q}{k} - 1) - (D_{i-1}+t_i)] - [\, N_{i-1} \cdot \frac{1}{W}(\frac{Q}{k} - 1) - D_{i-1} \,]$$
$$= \frac{1}{W}(\frac{Q}{k} - 1).$$

[Case 2]: The sweep-direction is changed after serving this request.

In this case, $N_i = 0$ and $D_i = 0$. $\Phi_i = 0$.

$$a_i = t_i + \Phi_i - \Phi_{i-1}$$
$$= t_i + [0] - [\, N_{i-1} \cdot \frac{1}{W}(\frac{Q}{k} - 1) - D_{i-1} \,]$$
$$= t_i + D_{i-1} - N_{i-1} \cdot \frac{1}{W}(\frac{Q}{k} - 1).$$

Since the minimum length of the waiting queue is W by assumption, the minimum number of requests served in one sweep is also W. That is, the sweep-direction cannot be changed when the number of requests served in the current sweep is less than W. Therefore, $N_{i-1} \geq (W-1)$. Then,

$$a_i \leq t_i + D_{i-1} - (W-1) \cdot \frac{1}{W}(\frac{Q}{k} - 1)$$
$$\leq [t_i + D_{i-1} - (\frac{Q}{k} - 1)] + \frac{1}{W}(\frac{Q}{k} - 1).$$

Moreover, since the maximum distance which the disk heads move in one sweep is $(Q/k - 1)$, $t_i + D_{i-1} \leq (Q/k - 1)$. Therefore,

$$a_i \leq \frac{1}{W}(\frac{Q}{k} - 1)$$

According to the above discussions, $A = \frac{1}{W}(\frac{Q}{k} - 1)$.

Let T be the total seek distance to serve a sequence of m requests. Then

$$T = \sum_{i=1}^{m} t_i$$
$$= \sum_{i=1}^{m} a_i + \Phi_0 - \Phi_m$$

$$\leq \frac{m}{W}(\frac{Q}{k}\text{-1}) + \Phi_0 - \Phi_m$$

since $a_i \leq A = \frac{1}{W}(\frac{Q}{k}\text{-1})$.

Since both N_0 and D_0 are zero, $\Phi_0 = 0$ and hence

$$T \quad \leq \frac{m}{W}(\frac{Q}{k}\text{-1}) - \Phi_m.$$

Moreover, if $N_m = 0$, then D_m=0 and Φ_m=0. Otherwise, $N_m \geq 1$, then $D_m \in [0, (\frac{Q}{k}\text{-1})]$ and $\Phi_m = N_m \cdot \frac{1}{W}(\frac{Q}{k}\text{-1}) - D_m \geq \frac{1}{W}(\frac{Q}{k}\text{-1}) - (\frac{Q}{k}\text{-1})$. The minimum of Φ_m is $(\frac{1}{W}\text{-1})\cdot(\frac{Q}{k}\text{-1})$. Accordingly,

$$T \quad \leq \frac{m}{W}(\frac{Q}{k}\text{-1}) - (\frac{1}{W}\text{-1})\cdot(\frac{Q}{k}\text{-1})$$

$$\leq \frac{m+W-1}{W}(\frac{Q}{k}\text{-1}).$$

Therefore, the amortized complexity of *PLAIN* is no greater than $\frac{m+W-1}{m \cdot W}(\frac{Q}{k}\text{-}1)$.

□

4. A Lower Bound of the Amortized Complexity of Restrictively Two-Headed Disk Systems

To explore how the best disk system with k restricted disk heads will behave, a lower bound of the amortized complexity of the restricted multi-headed disk systems is derived in the following theorem.

Theorem 2: (Lower Bound of Restricted Multi-Headed Disk Systems)

The amortized complexity of any restricted k-headed disk system must be no less than $\frac{1}{W}(\frac{Q}{k}\text{-}1)$.

Proof:

To prove the amortized complexity is lower-bounded by $\frac{1}{W}(\frac{Q}{k}\text{-}1)$, suppose there is a restricted k-headed disk system. Its head separation is d and amortized complexity is less than $\frac{1}{W}(\frac{Q}{k}\text{-}1)$.

Consider a sequence of requests $[(Q\text{-}1)^W, 0^W]^l$ for arbitrarily large l. Suppose the number of waiting requests in W at any time. This sequence should also be scheduled and processed by the sequence $[(Q\text{-}1)^W, 0^W]^l$. The length of this sequence is $2 \cdot l \cdot W$. Since the disk heads are initially placed on cylinders (0, d, ..., $(k\text{-}1)d$), the requests placed on cylinder *0* should be served by the first disk head and those placed on cylinder Q-1 should be served by the last head, i.e., the k-th disk head.

For the first W's requests placed on cylinder Q-1, the k disk heads should move from (0, d, ..., $(k\text{-}1)d$) to ($Q\text{-}1\text{-}(k\text{-}1)d$, $Q\text{-}1\text{-}(k\text{-}1)d+d$, ..., Q-1). Their total seek distance is $Q\text{-}1\text{-}(k\text{-}1)d$. By the same way, the total seek distance of the next W's requests placed on cylinder *0* is also $Q\text{-}1\text{-}(k\text{-}1)d$, and so on. The total seek distance of this sequence $[(Q\text{-}1)^W, 0^W]^l$ is then $2 \cdot (Q\text{-}1\text{-}(k\text{-}1)d) \cdot l$. Since the disk heads are restricted within the disk surface, d should be no greater than Q/k or the requests located in cylinders $Q\text{-}d$, $Q\text{-}d\text{+}1$, ..., and $d\text{-}1$ could not be served by any head. Therefore, $2 \cdot (Q\text{-}1\text{-}(k\text{-}1)d) \cdot l \geq 2 \cdot l(Q/k\text{-}1)$.

The average of the total seek distance is then $\frac{1}{W}(\frac{Q}{k}\text{-}1)$ (i.e., $\frac{2 \cdot l}{2 \cdot lW}(\frac{Q}{k}\text{-}1)$) which contradicts our previous assumption.

□

5. Conclusion

In this paper, we propose a restricted k-headed disk system and analyze it in amortized sense. According to our analysis, the amortized complexity of our restricted k-headed disk system *PLAIN* is $\frac{m+W-1}{m \cdot W}(\frac{Q}{k}\text{-}1)$. Moreover, the lower bound of the amortized complexity of a restricted k-headed disk system is $\frac{1}{W}(\frac{Q}{k}\text{-}1)$. Here Q is the total number of the cylinders and m is the number of requests in the considered sequence. Since both of these formulas are equal when $m \to \infty$, *PLAIN* is an optimal disk system with k restricted disk heads.

Based on our previous result in [2], SCAN is an optimal scheduling policy in one-headed disk system. Its amortized complexity is $\frac{m+W-1}{m \cdot W}Q$. Since $\frac{m+W-1}{m \cdot W} \cdot k(\frac{Q}{k}-1) \leq \frac{m+W-1}{m \cdot W}Q$, *PLAIN* performs more than k times as well as a system with one head.

Various authors have studied the two-headed disk systems based on amortized complexity [3] and some probability models [1,4,5]. They concluded that the optimum head separation is approximately $\frac{Q}{2}$, and a system with two heads performs more than twice as well as a system with a single head. Our result therefore matches their conclusion when $k = 2$.

Reference

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+This research is supported by the National Science Council, Taiwan R. O. C. under contract: NSC85-2213-E-167-006