The Mutually Independent Edge-Bipancyclic Property in Hypercube Graphs

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***Abstract*—A graph G is *edge-pancyclic* if each edge lies on cycles of all lengths. A bipartite graph is *edge-bipancyclic* if each edge lies on cycles of every even length from 4 to $|V(G)|$. Two cycles with the same length m, $C_1 = \langle u_1, u_2, \cdots, u_m, u_1\rangle$ and $C_2 = \langle v_1, v_2, \cdots, v_m, v_1\rangle$ passing through an edge (x, y) are *independent with respect to the edge* (x, y) if $u_1 = v_1 = x$, $u_m = v_m = y$ and $u_i \neq v_i$ for $2 \leq i \leq m-1$. Cycles with equal length $C_1, C_2, \cdots, C_n$ passing through an edge (x, y) are *mutually independent with respect to the edge* (x, y) if each pair of them are independent with respect to the edge (x, y). We propose a new concept called *mutually independent edge-bipancyclicity*. We say that a bipartite graph G is k-*mutually independent edge-bipancyclic* if for each edge $(x, y) \in E(G)$ and for each even length l, $4 \leq l \leq |V(G)|$, there are k cycles with the same length l passing through edge (x, y), and these k cycles are mutually independent with respect to the edge (x, y). In this paper, we prove that the hypercube Q_n is $(n-1)$-mutually independent edge-bipancyclic for $n \geq 4$.**

***Index Terms*—hypercube, bipancyclic, edge-bipancyclic, mutually independent.**

I. Introduction

For the graph definitions and notations we refer the reader to [1]. A graph is denoted by G with the vertex set $V(G)$ and the edge set $E(G)$. The simulation of one architecture by another is an important issue in interconnection networks. The problem of simulating one network by another is also called embedding problem. One particular problem of ring embedding deals with finding all the possible length of cycles in an interconnection network [2]–[4].

A path $P = \langle v_0, v_1, \cdots, v_m\rangle$ is a sequence of adjacent vertices. We also write $P = \langle v_0, \cdots, v_i, Q, v_j, \cdots, v_m\rangle$ where Q is a path $\langle v_i, \cdots, v_j\rangle$. A cycle $C = \langle v_0, v_1, \cdots, v_m, v_0\rangle$ is a sequence of adjacent vertices. The *length* of a path P is the number of edges in P. The *length* of a cycle C is the number of edges in C.

A path is a *hamitonian path* if it contains all the vertices of G. A graph G is *hamiltonian connected* if there exists a hamiltonian path between any two different vertices of G. A graph $G = (B \cup W, E)$ is *bipartite* if $V(G)$ is the union of two disjoint sets B and W such that every edge joins B with W. It is easy to see that any bipartite graph with at least three vertices is not hamiltonian connected. A bipartite graph G is *hamiltonian laceable* if there exists a hamiltonian path joining any two vertices from different partite sets. A graph G is *pancyclic* [1] if G includes cycles of all lengths. A graph G is called *edge-pancyclic* if each edge lies on cycles of all lengths. If these cycles are restricted to even length, G is called a *bipancyclic graph*. A bipartite graph is *edge-bipancyclic* [5] if each edge lies on cycles of every even length from 4 to $|V(G)|$. A graph is *panconnected* if, for any two different vertices x and y, there exists a path of length l joining x and y, for every l, $d_G(x, y) \leq l \leq |V(G)|-1$. The concept of panconnected graphs is proposed by Alavi and Williamson [6]. It is not hard to see that any bipartite graph with at least 3 vertices is not panconnected. Therefore, the concept of bipanconnected graphs is proposed. A bipartite graph is *bipanconnected* if, for any two different vertices x and y, there exists a path of length l joining x and y, for every l, $d_G(x, y) \leq l \leq |V(G)| - 1$ and $(l - d_G(x, y))$ being even. It is proved that the hypercube is bipanconnected [7].

We now introduce a relatively new concept. Two paths $P_1 = \langle u_1, u_2, \cdots, u_m \rangle$ and $P_2 = \langle v_1, v_2, \cdots, v_m \rangle$ from a to b are *independent* [8] if $u_1 = v_1 = a$, $u_m = v_m = b$, and $u_i \neq v_i$ for $2 \leq i \leq m-1$. Paths with equal length $P_1, P_2, \cdots, P_n$ from a to b are *mutually independent* [8] if every two different paths are independent. Two paths P_1 and P_2 are *fully independent* [9] if $u_i \neq v_i$ for all $1 \leq i \leq m$. Paths with equal length $P_1, P_2, \cdots, P_n$, are *mutually fully independent* if each pair of them are fully independent. Two cycles $C_1 = \langle u_1, u_2, \cdots, u_m, u_1 \rangle$ and $C_2 = \langle v_1, v_2, \cdots, v_m, v_1 \rangle$ passing through an edge (x, y) are *independent with respect to the edge* (x, y), if $u_1 = v_1 = x$, $u_m = v_m = y$ and $u_i \neq v_i$ for $2 \leq i \leq m-1$. Cycles with equal length $C_1, C_2, \cdots, C_n$ passing through an edge (x, y) are *mutually independent with respect to the edge* (x, y) if every two cycles are independent with respect to the edge (x, y).

An n-dimensional hypercube, denoted by Q_n, is a graph with 2^n vertices, and each vertex u can be distinctly labeled by an n-bit binary string, $u = u_{n-1}u_{n-2}...u_1u_0$. There is an edge between two vertices if and only if their binary labels differ in exactly one bit position. Let (u, v) be an edge in Q_n. If the binary labels of u and v differ in ith position, then the edge between them is said to be in ith dimension and the edge (u, v) is called an ith dimension edge. We use Q_n^0 to denote the subgraph of Q_n induce by $\{u \in V(Q_n) \mid u_i = 0\}$ and Q_n^1 the subgraph of Q_n induced by $\{u \in V(Q_n) \mid u_i = 1\}$. Q_n^0 and Q_n^1 are all isomorphic to Q_{n-1}. Q_n can be decomposed into Q_n^0 and Q_n^1 by dimension i, and Q_n^0 and Q_n^1 are $(n-1)$-dimensional subcubes of Q_n induced by the vertices with the ith bit position being 0 and 1 respectively. For each vertex u in Q_n^i, $i = 0, 1$, there is exactly one vertex in Q_n^{i-1}, denoted by $\bar{u}$, such that $(u, \bar{u})$ is an edge in Q_n. Saad and Schultz [10] proved Q_n is edge-bipancyclic in the sense that each edge lies on cycles of every even length from 4 to 2^n. Li et al. [7] considered an injured n-dimensional hypercube Q_n where each edge lies on cycles of every even length from 4 to 2^n with $n-2$ edge faults. Tsai [11] proved that such injured hypercube Q_n contains a cycle of every even length from 4 to 2^n, even if it has up to $(2n-5)$ edge faults with some specified conditions. Sun et al. [12] proved that the n-dimensional hypercube Q_n contains $n-1$ mutually independent hamiltonian paths between any vertex pair $\{x, y\}$, where x and y belong to different partite sets and $n \geq 4$. Let $|F|$ be the number of the faulty edges. Hsieh and Weng [13] showed that when $1 \leq |F| \leq n-2$, there exists $n-|F|-1$ mutually independent hamiltonian paths joining x to y in $Q_n - F$, where x and y belong to different partite sets.

We now introduce a new concept. We say that a bipartite graph G is *n-mutually independent edge-bipancyclic* if for each edge $(x, y) \in E(G)$, and for each even length l, $4 \leq l \leq |V(G)|$, there are n cycles with the same length l passing through edge (x, y), and these n cycles are mutually independent with respect to the edge (x, y). In this paper, we show that the hypercube has a stronger property of edge-bipancyclic property. We prove that an n-dimensional hypercube Q_n, for $n \geq 4$, is $(n-1)$-mutually independent edge-bipancyclic in the sense that each edge of Q_n lies on $n-1$ mutually independent cycles of every even length from 4 to 2^n. Our result strengthens a previous result of Saad and Schultz [10]. Because each vertex of the hypercube Q_n has exactly n edges incident with it, we can expect at most $n-1$ mutually independent cycles passing through edge (x, y). Therefore, the result "$n-1$" is tight.

II. Preliminaries

In order to prove our claim, we need the following previous results.

Lemma 1. *[14] The hypercube Q_n is hamiltonian laceable for every positive integer n.*

Lemma 2. *[7] The hypercube Q_n is bipanconnected for $n \geq 2$.*

The hypercube Q_n is known to be a bipartite graph. Let (B, W) be the vertex bipartition of Q_n. Edges $e_1, e_2, \cdots, e_n$ in a graph G are called *independent edges* if these edges are pairwise disjoint.

Lemma 3. *[12] Let $\{e_i \mid 1 \leq i \leq n-1\}$ be any $n-1$ independent edges of Q_n with $n \geq 2$ and $e_i = (b_i, w_i)$. Then there exist $n-1$ mutually fully independent hamiltonian paths $P_1^l, \cdots, P_{n-1}^l$ of Q_n such that P_i^l joins from b_i to w_i.*

Theorem 1. *[12] Let x and y be two vertices from different partite sets of Q_n, for $n \geq 4$. Then there exist $n-1$ mutually independent hamiltonian paths joining x to y.*

Theorem 2. *[15] Let F_v be a set of faulty vertices in Q_n. There exists a path of every odd length from 3 to $2^n - 2|F_v| - 1$ joining any two adjacent fault-free vertices in $Q_n - F_v$ even if $|F_v| \leq n-2$, where $n \geq 3$.*

Lemma 4. *[12] $Q_n - \{x, y\}$ is hamiltonian laceable, if x and y are any two vertices from different partite sets of Q_n with $n \geq 4$.*

III. Mutually independent edge-bipancyclic property of hypercubes

To prove our main result, we need the following lemmas, Lemma 5 to 7.

Lemma 5. *Let x and y be two vertices from different partite sets of Q_n with $n \geq 4$. There exists a path of every odd length from 1 to $2^n - 3$ joining any two adjacent fault-free vertices in $Q_n - \{x, y\}$.*

By Theorem 2 and Lemma 4, we can prove Lemma 5 easily.

Lemma 6. *Let e_1 and e_2 be two independent edges of Q_3, $e_i = (b_i, w_i)$ for $i = 1, 2$. Then Q_3 contains 2 mutually fully independent paths P_1^l and P_2^l with any odd length $l \leq 2^3$ such that P_i^l joins from b_i to w_i, $i = 1, 2$.*

Lemma 7. *Let $\{e_i \mid 1 \leq i \leq n-1\}$ be $n-1$ independent edges of Q_n with $n \geq 2$, $e_i = (b_i, w_i)$, $i = 1$ to $n-1$. Then there exist $n-1$ mutually fully independent paths $P_1^l, \cdots, P_{n-1}^l$ of Q_n with any odd length $l \leq 2^n - 1$ such that P_i^l joins from b_i to w_i, $i = 1$ to $n-1$.*

Proof: It is clear that the result holds for Q_2. We prove the statement by induction on n. By Lemma 6, the statement holds for $n = 3$. Suppose that the result holds for Q_{n-1}, for some $n \geq 4$. The hypercube Q_n has n dimensions, and there are only $n-1$ independent edges, so there is at least one dimension which does not contain any one of these $n-1$ independent edges. We can choose one of these dimensions to separate Q_n into two $(n-1)$-dimensional subcubes Q_n^0 and Q_n^1. We then prove the result by considering the following three cases.

Case 1. For odd length l and $1 \leq l \leq 2^{n-1} - 1$.

Case 1.1. Suppose that there are k independent edges in Q_n^0 with $1 \leq k \leq n-2$ and there are $n-k-1$ independent edges in Q_n^1. By induction hypothesis, the case is obvious.

Case 1.2. Without loss of generality, suppose that all the $n-1$ independent edges are in Q_n^0. By induction hypothesis, there exist $n-2$ mutually fully independent paths $P_1^l, \cdots, P_{n-2}^l$ of Q_n^0 with any odd length $l \leq 2^{n-1} - 1$ such that P_i^l joins from b_i to w_i for $1 \leq i \leq n-2$. By Lemma 2, there is a path R^m of Q_n^1 with odd length $1 \leq m \leq 2^{n-1} - 3$ joining $\bar{b}_{n-1}$ to $\bar{w}_{n-1}$. Let $P_{n-1}^l = \langle b_{n-1}, \bar{b}_{n-1}, R^m, \bar{w}_{n-1}, w_{n-1} \rangle$, then $3 \leq l \leq 2^{n-1} - 1$. Note that, b_{n-1} and w_{n-1} are adjacent vertices, so we obtain paths P_{n-1}^l for all odd lengths l, $1 \leq l \leq 2^{n-1} - 1$. Therefore, there are $n-1$ mutually fully independent paths $P_1^l, \cdots, P_{n-1}^l$ of Q_n with any odd length $l \leq 2^n - 1$ such that P_i^l joins from b_i to w_i, $i = 1$ to $n-1$.

Case 2. For odd length l and $2^{n-1} + 1 \leq l \leq 2^n - 3$.

Case 2.1. Suppose that there are k independent edges in Q_n^0 with $1 \leq k \leq n-2$ and there are $n-k-1$ independent edges in Q_n^1. By induction hypothesis, there exist k mutually fully independent paths $R_1, \cdots, R_k$ of Q_n^0 with length $2^{n-1} - 1$ such that R_i joins from b_i to w_i for $1 \leq i \leq k$. We let $R_i = \langle b_i, u_i, v_i, Z_i, w_i \rangle$ for $1 \leq i \leq k$. According to induction hypothesis, there exist k mutually fully independent paths $T_1^{l'}, \cdots, T_k^{l'}$ of Q_n^1 with any odd length $l' \leq 2^{n-1} - 3$ such that $T_i^{l'}$ joins from $\bar{u}_i$ to $\bar{v}_i$ for $1 \leq i \leq k$. Therefore, $P_i^l = \langle b_i, u_i, \bar{u}_i, T_i^{l'}, \bar{v}_i, v_i, Z_i, w_i \rangle$ with $2^{n-1} + 1 \leq l \leq 2^n - 3$ for $1 \leq i \leq k$. Again by induction hypothesis, there exist $n-k-1$ mutually fully independent paths $R_{k+1}, \cdots, R_{n-1}$ of Q_n^1 with length $2^{n-1} - 1$ such that R_i joins from b_i to w_i for $k+1 \leq i \leq n-1$. We let $R_i = \langle b_i, u_i, v_i, Z_i, w_i \rangle$ for $k+1 \leq i \leq n-1$. By induction hypothesis, there exist $n-k-1$ mutually fully independent paths $T_{k+1}^{l'}, \cdots, T_{n-1}^{l'}$ of Q_n^0 with any odd length $l' \leq 2^{n-1} - 3$ such that $T_i^{l'}$ joins from $\bar{u}_i$ to $\bar{v}_i$ for $k+1 \leq i \leq n-1$. Therefore, $P_i^l = \langle b_i, u_i, \bar{u}_i, T_i^{l'}, \bar{v}_i, v_i, Z_i, w_i \rangle$ with $2^{n-1} + 1 \leq l \leq 2^n - 3$ for $k+1 \leq i \leq n-1$. Hence, there are $n-1$ mutually fully independent paths $P_1^l, \cdots, P_{n-1}^l$ of Q_n with any odd length $2^{n-1} + 1 \leq l \leq 2^n - 3$ such that P_i^l joins from b_i to w_i.

Case 2.2. Without loss of generality, suppose that all the $n-1$ independent edges are in Q_n^0. By induction hypothesis, there exist $n-2$ mutually fully independent paths $R_1, \cdots, R_{n-2}$ of Q_n^0 with length $2^{n-1}-1$ such that R_i joins from b_i to w_i for $1 \leq i \leq n-2$. We let $R_i = \langle b_i, Z_i, u_i, v_i, z_i, w_i \rangle$. Again by induction hypothesis, there exist $n-2$ mutually fully independent paths $T_1^{l'}, \cdots, T_{n-2}^{l'}$ of Q_n^1 with any odd length $l' \leq 2^{n-1}-3$ such that $T_i^{l'}$ joins from $\bar{u}_i$ to $\bar{v}_i$ for $1 \leq i \leq n-2$. Therefore, $P_i^l = \langle b_i, Z_i, u_i, \bar{u}_i, T_i^{l'}, \bar{v}_i, v_i, z_i, w_i \rangle$ with any odd length $2^{n-1}+1 \leq l \leq 2^n-3$ for $1 \leq i \leq n-2$. By Lemma 2, there exists a path R_{n-1} of Q_n^1 with length $2^{n-1}-3$ joining $\bar{b}_{n-1}$ to $\bar{w}_{n-1}$. We let $R_{n-1} = \langle \bar{b}_{n-1}, Z_{n-1}, u_{n-1}, v_{n-1}, \bar{w}_{n-1} \rangle$. By Lemma 5, there exists a path $T_{n-1}^{l'}$ with every odd length $1 \leq l' \leq 2^{n-1}-3$ joining $\bar{u}_{n-1}$ to $\bar{v}_{n-1}$ in $Q_n^0 - \{b_{n-1}, w_{n-1}\}$. Therefore, $P_{n-1}^l = \langle b_{n-1}, \bar{b}_{n-1}, Z_{n-1}, u_{n-1}, \bar{u}_{n-1}, T_{n-1}^{l'}, \bar{v}_{n-1}, v_{n-1}, \bar{w}_{n-1}, w_{n-1} \rangle$ with $2^{n-1}+1 \leq l \leq 2^n-3$. So, there are $n-1$ mutually fully independent paths $P_1^l, \cdots, P_{n-1}^l$ of Q_n with any odd length $2^{n-1}+1 \leq l \leq 2^n-3$ such that P_i^l joins from b_i to w_i.

Case 3. For odd length l and $l = 2^n - 1$. This case is proved by Lemma 3.

By Case 1 Case 2 and Case 3, the proof is complete. ■

We now prove our main result by induction.

Lemma 8. *The hypercube Q_4 is 3-mutually independent edge-bipancyclic.*

Theorem 3. *The hypercube Q_n is $(n-1)$-mutually independent edge-bipancyclic for $n \geq 4$.*

Proof: Let (u, v) be an edge in Q_n, $n \geq 4$. We prove the statement by induction on n. By Lemma 8, the statement holds for $n = 4$. Suppose that the result holds for Q_{n-1}, $n \geq 5$. We may choose a dimension to divide the hypercube Q_n into two subcubes Q_n^0 and Q_n^1 so that the edge (u, v) is in Q_n^0. According to the length l of the cycles, we divide the proof into the following three cases. In each case, the length l is assumed to be an even number. We shall find $n-1$ mutually independent cycles with length l passing through edge (u, v).

Case 1. For even length l and $4 \leq l \leq 2^{n-1}$.

By induction hypothesis, there exist $n-2$ mutually independent cycles with respect to the edge (u, v), $C_1^k, \cdots, C_{n-2}^l$ with any even length $4 \leq l \leq 2^{n-1}$ in Q_n^0. By Lemma 2, there is a path P^k of Q_n^1 with any odd length $1 \leq k \leq 2^{n-1}-3$ joining $\bar{u}$ to $\bar{v}$. Then we have $C_{n-1}^l = \langle u, \bar{u}, P^k, \bar{v}, v, u \rangle$ with any even length $4 \leq l \leq 2^{n-1}$. Therefore, there exist $n-1$ mutually independent cycles with respect to the edge (u, v), $C_1^l, \cdots, C_{n-1}^l$ with every even length $4 \leq l \leq 2^{n-1}$.

Case 2. For even length l and $2^{n-1}+2 \leq l \leq 2^n-2$. By induction hypothesis, there exist $n-2$ mutually independent cycles with respect to the edge (u, v), $R_1, \cdots, R_{n-2}$ with length 2^{n-1} of Q_n^0. We let $R_i = \langle u, x_i, y_i, z_i, \cdots, v, u \rangle$ for $1 \leq i \leq n-2$. By Lemma 7, for any given odd length $k \leq 2^{n-1}-3$ there exist $n-2$ mutually fully independent paths $P_1^k, \cdots, P_{n-2}^k$ all with the same length k, such that P_i^k joins from $\bar{y}_i$ to $\bar{z}_i$ for $1 \leq i \leq n-2$. We let $C_i^l = \langle u_i, x_i, y_i, \bar{y}_i, P_i^k, \bar{z}_i, z_i, R_i, v_i, u_i \rangle$. Then C_i^l, $i = 1$ to $n-2$ are with any even length l, where $2^{n-1}+2 \leq l \leq 2^n-2$. By Lemma 1, there exists a hamiltonian path P' of Q_n^1 joining $\bar{u}$ to $\bar{v}$. Let $P' = \langle \bar{u}, y_{n-1}, z_{n-1}, T, \bar{v} \rangle$. By Lemma 5, there exists a path $U^{k'}$ with every odd length $1 \leq k' \leq 2^{n-1}-3$ joining $\bar{y}_{n-1}$ to $\bar{z}_{n-1}$ in $Q_n^0 - \{u, v\}$. We let $C_{n-1}^l = \langle u, \bar{u}, y_{n-1}, \bar{y}_{n-1}, U^{k'}, \bar{z}_{n+1}, z_{n+1}, T, \bar{v}, v \rangle$ with any even length l, $2^{n-1}+2 \leq l \leq 2^n-2$. Hence, there exist $n-1$ mutually independent cycles with respect to edge (u, v), $C_1^l, \cdots, C_{n-1}^l$ with any even length $4 \leq l \leq 2^{n-1}$.

Case 3. For even length l and $l = 2^n$. This case is proved by Theorem 1.

By Case 1, Case 2 and Case 3, we complete the proof. ■

IV. Conclusion

In [1], the author introduced a popular property called the *pancyclicity*. A stronger property is *edge-bipancyclicity* which was proposed by Mitchem and Schmeichel in [5]. Another interesting property is the *mutually independent paths*. Sun et al. [12] proved that the n-dimensional hypercube graph contains $n-1$ mutually independent hamiltonian paths between any vertex pair $\{x, y\}$, where x and y belong to different partite sets and $n \geq 4$. In this paper, we combine the two properties, edge-bipancyclicity and mutually independent paths, into a new stronger property called *mutually independent edge-bipancyclic property*, and show that the hypercube Q_n is $(n-1)$-mutually independent edge-pancyclic for $n \geq 4$. Our result also strengthens a previous result of Saad and Schultz [10], in the sense that the hypercube Q_n is not only edge-bipancyclic but also mutually independent edge-pancyclic.

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