Pancycle problem of crossed cube with conditional fault

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ABSTRACT

The crossed cube, which is a variation of the hypercube, possesses some properties superior to the hypercube. In this paper, with the assumption that each node is incident with at least two fault-free links, fault-free cycles of all possible lengths are constructed in an n-dimensional crossed cube with up to $2n-5$ link faults. The result is optimal with respect to the number of link faults tolerated.

1: Introduction

The crossed cube has been studied extensively in the literature [3], [4], [5], [6], [9], [11], [12], [15]. It was demonstrated in [11] that a (2^n–1)-node complete binary tree can be embedded into a 2^n- node crossed cube with dilation 1. The dilation will go up to 2 if the same tree is embedded into a 2^n-node hypercube (see [14]). The connectivity of an *n*-dimensional crossed cube was shown to be *n* in [12]. In [3], the *n*-wide diameter and the (*n*–1)-fault diameter of an *n*-dimensional crossed cube were computed, which are $\lceil n/2 \rceil+2$. Moreover, the crossed cube was shown to be pancyclic, while the hypercube is not. It was further shown in [9], [15] that an *n*-dimensional crossed cube is (*n*–2)-Hamiltonian, (*n*–3)-Hamiltonian connected, and (*n*–2)-fault tolerant pancyclic.

Random faults were assumed in [9], [15]. That is, faults may happen anywhere in the crossed cube without any restriction. This is the main reason that an *n*-dimensional crossed cube can tolerate no more than *n*–2 faults, while retaining a fault-free Hamiltonian cycle. In [10], conditional faults are assumed, i.e., the fault distribution is restricted. It was shown that if each node is incident with at least two fault-free links, then a fault-free Hamiltonian cycle can be found in an *n*-dimensional crossed cube, even if there are up to 2*n*–5 link faults. In this paper, we will show that under the same assumption, fault-free cycles with length range from 4 to 2^n also can be found in a crossed cube with up to 2*n*–5 link faults. It was shown in [10] that the assumption is practically meaningful by evaluating its probability of occurrence, which is very close to 1, even if *n* is small. The result is optimal, because there are distributions of 2*n*–4 link faults which can prevent a fault-free Hamiltonian cycle in the crossed cube. It was shown in [2]([1]) that if each node of an *n*-dimensional hypercube (*k*-ary hypercube) is incident with at least two fault-free links, then it contains a fault-free Hamiltonian cycle, even if there are 2*n*–5 (4*n*–5) link faults. In [2], the problem of determining if a hypercube with an arbitrary number of link faults contains a Hamiltonian cycle was also shown to be NP-complete. In [13], with the same assumption of conditional link faults, it was shown that an *n*-dimensional hypercube remains strongly (fault- free) Hamiltonian laceable, even if there are 2*n*–5 link faults.

2: Preliminaries

A network such as the crossed cube is conveniently represented with an undirected graph, where the vertices (edges) of the graph denote the nodes (links) of the network. Throughout this paper, vertex and node, edge and link, and graph and network are used interchangeably. We use CQ_n to denote an *n*-dimensional crossed cube. Each node of CQ_n is uniquely identified with an *n*-bit sequence. CQ_1 and CQ_2 are the same as a one-dimensional hypercube and two-dimensional hypercube, respectively.

For $n \geq 3$, CQ_n can be obtained by joining two CQ_{n-1}'s, denoted by CQ_{n-1}^0 and CQ_{n-1}^1, with 2^{n-1} links, where each node of CQ_{n-1}^0 (CQ_{n-1}^1) is preceded with a bit 0 (1). A node $u = 0u_{n-2}u_{n-3}\ldots u_0 \in CQ_{n-1}^0$ is connected to a node $v = 1v_{n-2}v_{n-3}\ldots v_0 \in CQ_{n-1}^1$ if and only if $(u_{2i+1}u_{2i}, v_{2i+1}v_{2i}) \in \{(00, 00), (10, 10), (01, 11), (11, 01)\}$ for all $0 \leq i \leq \lfloor (n-1)/2 \rfloor - 1$ and $u_{n-2} = v_{n-2}$ if *n* is even.

Formally, CQ_n can be defined as follows, where $u_{2i+1}u_{2i} \sim v_{2i+1}v_{2i}$ denotes $(u_{2i+1}u_{2i}, v_{2i+1}v_{2i}) \in \{(00, 00), (10, 10), (01, 11), (11, 01)\}$.

Definition 1.[4] The node set of CQ_n is $\{v_{n-1}v_{n-2}\ldots v_0 \mid v_i \in \{0, 1\}$ for all $0 \leq i \leq n-1\}$. Two nodes $u = u_{n-1}u_{n-2}\ldots u_0$ and $v = v_{n-1}v_{n-2}\ldots v_0$ of CQ_n are adjacent if and only if there exists $0 \leq d \leq n-1$, satisfying the following four conditions:

(1) $u_{2i+1}u_{2i} \sim v_{2i+1}v_{2i}$ for all $0 \le i \le \lfloor d/2 \rfloor - 1$, if $d \ge 2$;
(2) $u_{d-1} = v_{d-1}$, if d is odd;
(3) $u_d = \overline{v_d}$ ($\overline{v_d}$ is the complement of v_d);
(4) $u_{n-1}u_{n-2}\ldots u_{d+1} = v_{n-1}v_{n-2}\ldots v_{d+1}$, if $d < n-1$.

When $d \ge 2$, the link (u, v) above connects CQ_d^0 with CQ_d^1, which is referred to as a *d-link*. When $d = 0$, the link (u, v) has $u_0 = \overline{v_0}$ and $u_i = v_i$ for $i \in \{1, 2, \ldots, n-1\}$; when $d = 1$, the link (u, v) has $u_1 = \overline{v_1}$ and $u_i = v_i$ for $i \in \{0, 1, \ldots, n-1\} - \{1\}$. The two links are referred to as 0-*link* and 1-*link*, respectively. Each node of CQ_n is incident with n links that are 0-link, 1-link, …, (n–1)-link, respectively.

A path (cycle) in a graph G is called a *Hamiltonian path* (*cycle*) if it contains every vertex of G exactly once. The number of edges incident with a vertex v in G is called the *degree* of v. Throughout this paper, we use $V(G)$ ($E(G)$) to denote the vertex set (edge set) of G and $G - V'$ ($G - E'$) to denote the graph that results by removing V' (E') from G, where $V' \subseteq V(G)$ and $E' \subseteq E(G)$. We also use $P_{x,y}$ to denote a path from node x to node y, $\delta(G)$ to denote the minimum node degree of G, and $u^{(d)}$ to denote the node that is connected to u by a d-link of CQ_n. When $x = y$, $P_{x,y}$ denotes a cycle. A cycle of length l is referred to as an *l-cycle*.

Lemma 1.[9] Suppose that $E' \subset E(CQ_n)$ and $|E'| \le n-2$, where $n \ge 3$. Then, there exists a Hamiltonian cycle in $CQ_n - E'$.

Lemma 2.[9] Suppose that $E' \subset E(CQ_n)$, $V' \subset V(CQ_n)$, and $|E'| + |V'| \le n-3$, where $n \ge 3$. Then, for every two distinct nodes u, v in $V(CQ_n) - V'$, there exists a Hamiltonian path between u and v in $CQ_n - V' - E'$.

Lemma 3.[10] Suppose that $E' \subset E(CQ_n)$, $|E'| \le 2n-5$, and $\delta(CQ_n - E') \ge 2$, where $n \ge 3$. Then, there is a Hamiltonian cycle in $CQ_n - E'$.

Lemma 4.[15] Suppose that $E' \subset E(CQ_n)$ and $|E'| \le n-2$, where $n \ge 3$. Then, for $4 \le l \le 2^n$, there is an l-cycle in $CQ_n - E'$.

Lemma 5.[10] Suppose that u, v, x and y are four distinct nodes of CQ_n, where $n \ge 4$. Then, there are a $P_{u,v}$ and a $P_{x,y}$ so that $V(P_{u,v}) \cap V(P_{x,y}) = \varnothing$ and $V(P_{u,v}) \cup V(P_{x,y}) = V(CQ_n)$.

Lemma 6.[12] Suppose that (s, t) is a d-link of CQ_n, where d is odd or $d = n-2$ and $n \ge 2$. Then, $(s^{(n-1)}, t^{(n-1)})$ is also a d-link of CQ_n.

Clearly, when $d \ne n-1$, the four nodes s, $s^{(n-1)}$, $t^{(n-1)}$ and t mentioned in Lemma 6 form a 4-cycle. In particular, when $d = n-2$, such a 4-cycle is referred to as a *crossed* 4-*cycle*. It is not difficult to see that each node of CQ_n is contained in a unique crossed 4-cycle. Thus, there are 2^{n-2} mutually disjoint crossed 4-cycles in CQ_n.

Lemma 7. Suppose that (s, t) is a 0-link of CQ_n, where $n \ge 3$. Then, s, t, $t^{(n-1)}$, $(s^{(n-1)})^{(0)}$ and $s^{(n-1)}$ form a 5-cycle.
Proof. Clearly, (s, t), $(t, t^{(n-1)})$, $((s^{(n-1)})^{(0)}, s^{(n-1)})$ and $(s^{(n-1)}, s)$ are all links of CQ_n. We only need to show $(t^{(n-1)}, (s^{(n-1)})^{(0)}) \in E(CQ_n)$. Assume $s = s_{n-1}s_{n-2}\ldots s_1s_0$, and so $t = s_{n-1}s_{n-2}\ldots s_1\overline{s_0}$. Since $s^{(n-1)}$ and $t^{(n-1)}$ differ at bit positions 0 and 1, $(t^{(n-1)}, (s^{(n-1)})^{(0)})$ is a 1-link of CQ_n.

Similarly, s, t, $t^{(n-1)}$, $(t^{(n-1)})^{(0)}$ and $s^{(n-1)}$ also form a 5-cycle in CQ_n, where (s, t) is a 0-link. The two 5-cycles have four nodes in common; the other two distinct nodes are $(s^{(n-1)})^{(0)}$ and $(t^{(n-1)})^{(0)}$, which differ at bit positions 0 and 1. The 5-cycle whose distinct node (i.e., $(s^{(n-1)})^{(0)}$ or $(t^{(n-1)})^{(0)}$) has the rightmost bit 1 is referred to as a *crossed* 5-*cycle*. Since there are a total of 2^{n-1} 0-links, CQ_n contains 2^{n-1} crossed 5-cycles. □

Lemma 8.[10] Suppose that $E' \subset E(CQ_n)$ and $|E'| \le n-2$. If $(u, v) \notin E'$ is an $(n-1)$-link or a d-link for some odd d, then there exists a Hamiltonian cycle in $CQ_n - E'$ that contains (u, v).

Lemma 9.[7] Suppose that (u, v) is an $(n-1)$-link of CQ_n, where $n \ge 3$. Then, for $4 \le l \le 2^n$, there is an l-cycle in CQ_n that contains (u, v).

Lemma 10. Each link of CQ_n is contained in at most two crossed 5-cycles.
Proof. Consider an arbitrary d-link (x, y) of CQ_n, where $0 \le d \le n-1$. Recall that a crossed 5- cycle can be expressed as s, t, $t^{(n-1)}$, $(s^{(n-1)})^{(0)}$, $s^{(n-1)}$ or s, t, $t^{(n-1)}$, $(t^{(n-1)})^{(0)}$, $s^{(n-1)}$, depending on which of $(s^{(n-1)})^{(0)}$ and $(t^{(n-1)})^{(0)}$ has the rightmost bit 1. For convenience, we use z_1, z_2, z_3, z_4, z_5 to denote a crossed 5-cycle, where z_4 is the distinct node with the rightmost bit 1. Moreover, (z_1, z_2) and (z_4, z_5) are two 0-links, (z_2, z_3) and (z_5, z_1) are two $(n-1)$-links, and (z_3, z_4) is a 1-link. It is easy to see that the rightmost bits of z_1, z_2, z_3, z_4 and z_5 are 0, 1, 1, 1 and 0, respectively.

When $d \notin \{0, 1, n-1\}$, (x, y) is not contained in any crossed 5-cycle. Let x_0 and y_0 be the rightmost bits of x and y, respectively. When $d = 0$, we have $x_0 \ne y_0$. Without loss of generality, we assume $x_0 = 0$ and $y_0 = 1$. Then, (x, y) is contained in two crossed 5-cycles with $(z_1, z_2) = (x, y)$ and $(z_4, z_5) = (y, x)$, respectively. When $d = 1$, we have $x_0 = y_0$. If $x_0 = y_0 = 1$, then (x, y) is contained in two crossed 5-cycles with $(z_3, z_4) = (x, y)$ and $(z_3, z_4) = (y, x)$, respectively. If $x_0 = y_0 = 0$, then (x, y) is not contained in any crossed 5-cycle. When $d = n-1$, we have $x_0 = y_0$. If $x_0 = y_0 = 1$ ($x_0 = y_0 = 0$), then (x, y) is contained in two crossed 5-cycles with $(z_1, z_5) = (x, y)$ and $(z_1, z_5) = (y, x)$ ($(z_2, z_3) = (x, y)$ and $(z_2, z_3) = (y, x)$), respectively. □

Lemma 11. Suppose that $E' \subset E(CQ_n)$ and $|E'| \le n-3$, where $n \ge 3$. For every two distinct nodes u, v of CQ_n, there exists a $P_{u,v}$ of length $2^n - 2$ in $CQ_n - E'$.

Proof: When $|E'| \leq n-4$, a node $w \notin \{u, v\}$ of CQ_n is arbitrarily selected. By Lemma 2, there exists a $P_{u,v}$ of length 2^n-2 in $CQ_n - \{w\} - E'$. When $|E'| = n-3$, a node $w \notin \{u, v\}$ of CQ_n is selected such that there is a link $(w, z) \in E'$. There exists a $P_{u,v}$ of length 2^n-2 in $CQ_n - \{w\} - (E' - \{(w, z)\})$, by Lemma 2 again. □

Lemma 12. Suppose that $E' \subset E(CQ_n)$ and $|E'| \leq n-2$, where $n \geq 4$. If $(u, v) \notin E'$ is an $(n-1)$-link, then there exists a (2^n-1)-cycle in $CQ_n - E'$ that contains (u, v).

Proof: Partition E' into E_0, E_1 and E_c, where $E_0 = E' \cap E(CQ^0_{n-1})$, $E_1 = E' \cap E(CQ^1_{n-1})$, and $E_c = E' \cap \{(x, y) \mid x \in V(CQ^0_{n-1})$ and $y \in V(CQ^1_{n-1})\}$. Without loss of generality, assume that $u \in V(CQ^0_{n-1})$, $v \in V(CQ^1_{n-1})$, and $|E_0| \geq |E_1|$. When $n = 4$, this lemma can be easily verified by a computer program, which performs an exhaustive search on CQ_4. For $n \geq 5$, three cases are discussed below.

Case 1. $|E_0| \leq n-3$. When $|E_0| \leq n-4$, select $(s, s^{(n-1)}) \notin E'$ so that $(s, s^{(n-1)}) \neq (u, v)$. Without loss of generality, assume $s \in V(CQ^0_{n-1})$. By Lemma 2, there exists a Hamiltonian path between s and u in $CQ^0_{n-1} - E_0$. By Lemma 11, there exists a $P_{v,s^{(n-1)}}$ of length $2^{n-1}-2$ in $CQ^1_{n-1} - E_1$. The desired (2^n-1)-cycle in $CQ_n - E'$ can be constructed. When $|E_0| = n-3$, the cycle needs a modification. The Hamiltonian path in $CQ^0_{n-1} - E_0$ is not available again. Instead, a Hamiltonian cycle in $CQ^0_{n-1} - E_0$ is constructed by Lemma 1. Moreover, since $|E_c| \leq 1$, there exists $(s, s^{(n-1)}) \notin E'$ so that (s, u) is a link of the Hamiltonian cycle.

Case 2. $|E_0| = n-2$. Arbitrarily select a link $(x, y) \in E_0$. By Lemma 1, there exists a Hamiltonian cycle in $CQ^0_{n-1} - (E_0 - \{(x, y)\})$. If the Hamiltonian cycle does not contain (x, y), then select $(s, s^{(n-1)}) \notin E'$ so that (s, u) is a link of the Hamiltonian cycle. By Lemma 11, there exists a $P_{v,s^{(n-1)}}$ of length $2^{n-1}-2$ in $CQ^1_{n-1} - E_1$. The desired (2^n-1)-cycle in $CQ_n - E'$ can be obtained.

If the Hamiltonian cycle contains (x, y), then two situations, $u \in \{x, y\}$ or $u \notin \{x, y\}$, are discussed. When $u \in \{x, y\}$, we assume $u = x$ without loss of generality. By Lemma 11, there exists a $P_{v,y^{(n-1)}}$ of length $2^{n-1}-2$ in $CQ^1_{n-1} - E_1$. The desired (2^n-1)-cycle in $CQ_n - E'$ can be obtained (replacing s with y). When $u \notin \{x, y\}$, select a node $s \notin \{x, y\}$ such that (s, u) is a link of the Hamiltonian cycle. Also, select a node $t \notin \{u, s\}$ such that (t, x) or (t, y) is a link of the Hamiltonian cycle. By Lemma 5, there exist a $P_{v,s^{(n-1)}}$ and a $P_{t^{(n-1)},y^{(n-1)}}$ satisfying $V(P_{v,s^{(n-1)}}) \cap V(P_{t^{(n-1)},y^{(n-1)}}) = \varnothing$ and $V(P_{v,s^{(n-1)}}) \cup V(P_{t^{(n-1)},y^{(n-1)}}) = V(CQ^1_{n-1})$. The desired (2^n-1)-cycle in $CQ_n - E'$ can be constructed. □

3: Fault-Free Cycles of All Possible Lengths

It was shown in [15] that CQ_n with $n-2$ random link faults contains cycles of lengths ranging from 4 to 2^n (no 3-cycle in CQ_n [13]). In this section, we show that CQ_n can tolerate up to $2n-5$ link faults, while retaining these cycles, under the assumption that each node is incident with at least two fault-free links.

Theorem 1. Suppose that $E' \subset E(CQ_n)$, $|E'| \leq 2n-5$, and $\delta(CQ_n - E') \geq 2$, where $n \geq 3$. Then, there are cycles of lengths ranging from 4 to 2^n in $CQ_n - E'$.

Proof. For $n = 3$, the correctness of the theorem can be assured by Lemma 4. So, we assume $n \geq 4$. We prove by induction that there are cycles of lengths ranging from 4 to 2^n in $CQ_n - E'$ that each contain at least two $(n-1)$-links. When $n = 4$, cycles of lengths ranging from 4 to 16 that each contain at least two 3-links can be constructed in $CQ_4 - E'$, which can be verified by a computer program. We assume that it also holds for $n = k \geq 4$, i.e., there are cycles of lengths ranging from 4 to 2^k in $CQ_k - E'$ that each contain at least two $(k-1)$-links. In the rest of the proof, the situation of $n = k+1$ is discussed.

Partition E' into E_0, E_1 and E_c, where $E_0 = E' \cap E(CQ^0_k)$, $E_1 = E' \cap E(CQ^1_k)$, and $E_c = E' \cap \{(x, y) \mid x \in V(CQ^0_k)$ and $y \in V(CQ^1_k)\}$. Without loss of generality, we assume $|E_0| \geq |E_1|$. Since $|E_0| + |E_1| + |E_c| \leq 2k-3$, we have $|E_1| \leq k-2$. Recall that there are 2^{k-1} mutually disjoint crossed 4-cycles and 2^k crossed 5-cycles in CQ_{k+1}. Moreover, they each contain two k-links, and Lemma 10 assures that each link of CQ_{k+1} is contained in at most two crossed 5-cycles. Since $2^{k-1} > 2k-3$ ($2^k > 2 \times (2k-3)$) for $k \geq 4$, there are crossed 4-cycles (crossed 5-cycles) in $CQ_{k+1} - E'$. According to Lemma 3, there are 2^{k+1}-cycles in $CQ_{k+1} - E'$. In the following, l-cycles for $6 \leq l < 2^{k+1}$ in $CQ_{k+1} - E'$ that each contain at least two k-links are constructed with two cases.

Case 1. $6 \leq l \leq 2^k + 2$. Define $S = \{(u, v) \mid (u, v)$ is a $(k-1)$-link of CQ_{k+1}, $(u, v) \notin E'$, and $\{(u, u^{(k)}), (v, v^{(k)}), (u^{(k)}, v^{(k)})\} \cap E' \neq \varnothing\}$. Recall that $(u^{(k)}, v^{(k)})$ is a $(k-1)$-link of CQ_{k+1} by Lemma 6, and (u, v), $(u, u^{(k)})$, $(v, v^{(k)})$, $(u^{(k)}, v^{(k)})$ constitute a crossed 4-cycle in CQ_{k+1}. Suppose that (s, t)

$\notin E'$. is a $(k-1)$-link of CQ_{k+1}. Then, $\{(s, s^{(k)}), (t, t^{(k)}), (s^{(k)}, t^{(k)})\} \cap E' \neq \emptyset$ if and only if $(s, t) \in S$. So, if $(s, t) \notin S$, then $(s, s^{(k)}), (t, t^{(k)}), (s^{(k)}, t^{(k)}) \notin E'$.

Let $S_0 = S \cap E(CQ_k^0)$ and $S_1 = S \cap E(CQ_k^1)$. Suppose $(s, t) \in S_1$. Then, $(s, s^{(k)}) \in E'$ or $(t, t^{(k)}) \in E'$ or $(s^{(k)}, t^{(k)}) \in E'$, where $(s, s^{(k)}), (t, t^{(k)}) \in E_c$ and $(s^{(k)}, t^{(k)}) \in E_0$. It means that each link $(s, t) \in S_1$ induces at least one link $((s, s^{(k)})$ or $(t, t^{(k)})$ or $(s^{(k)}, t^{(k)}))$ in $E_0 \cup E_c$. Similarly, each link $(s, t) \in S_0$ induces at least one link in $E_1 \cup E_c$. Since no two distinct links in S_1 induce the same link in $E_0 \cup E_c$, we have $|S_1| \le |E_0 \cup E_c| = |E_0| + |E_c|$, which further implies $|S_1 \cup E_1| = |S_1| + |E_1| \le |E_0| + |E_1| + |E_c| \le 2k - 3$. Since $|E_1| \le k-2$ and each node in CQ_k^1 is incident with at most one link in S_1, we have $\delta(CQ_k^1 - (S_1 \cup E_1)) \ge 1$. Two subcases are discussed below.

Case 1.1. $|S_1 \cup E_1| \le 2k-4$. When $\delta(CQ_k^1 - (S_1 \cup E_1)) = 1$, there is exactly one node of degree one in $CQ_k^1 - (S_1 \cup E_1)$, for otherwise $|S_1 \cup E_1| \ge 2k-3$, a contradiction. Suppose that x is the node of degree one in $CQ_k^1 - (S_1 \cup E_1)$. Then, $(x, x^{(k-1)}) \in S_1$. By the induction hypothesis, there are cycles of lengths ranging from 4 to 2^k in $CQ_k^1 - (S_1 \cup E_1 - \{(x, x^{(k-1)})\})$ that each contain at least two $(k-1)$-links. Let C denote any of these cycles, and $(s, t) \neq (x, x^{(k-1)})$ be a $(k-1)$-link of C. Since $(s, t) \notin S_1$, we have $(s, s^{(k)}), (t, t^{(k)}), (s^{(k)}, t^{(k)}) \notin E'$. A $(|C|+2)$-cycle in $CQ_{k+1} - E'$ that contains two k-links, where $|C|$ is the length of C, can be constructed

Then, we assume $\delta(CQ_k^1 - (S_1 \cup E_1)) \ge 2$. l-cycles for $6 \le l \le 2^k + 2$ in $CQ_{k+1} - E'$ that each contain two k-links can be obtained, which is similar to the situation when $\delta(CQ_k^1 - (S_1 \cup E_1)) = 1$.

Case 1.2. $|S_1 \cup E_1| = 2k-3$. Since $|E_0| + |E_1| + |E_c| \le 2k-3 = |S_1 \cup E_1| = |S_1| + |E_1|$, we have $|E_0| + |E_c| \le |S_1|$. Recall that $|S_1| \le |E_0| + |E_c|$, and each link in S_1 induces at least one link in $E_0 \cup E_c$. So, $|S_1| = |E_0| + |E_c|$, and there is a one-to-one correspondence between S_1 and $E_0 \cup E_c$. Also notice that the image of each link in S_1 under the one-to-one correspondence is a $(k-1)$-link if it is located in E_0 and is a k-link if it is located in E_c. It follows that all links in E_0 are $(k-1)$-links. Moreover, if $(s, t) \in E_1$ is a $(k-1)$-link, then $(s, s^{(k)}), (t, t^{(k)}), (s^{(k)}, t^{(k)}) \notin E'$ for the following reason. Suppose conversely $(s^{(k)}, t^{(k)}) \in E'$, without loss of generality. The fact that $(s^{(k)}, t^{(k)}) \in E_0$ and $(s, t) \notin S_1$ contradicts the one-to-one correspondence between S_1 and $E_0 \cup E_c$.

When $|E_1| = 0$, arbitrarily pick a crossed 4-cycle in $CQ_{k+1} - E'$, and let $(s, t) \in E(CQ_k^1)$ be the $(k-1)$-link contained in the crossed 4-cycle. By Lemma 9, there are cycles of lengths ranging from 4 to 2^k in CQ_k^1 that each contain (s, t). Let C denote any of these cycles. A $(|C|+2)$-cycle in $CQ_{k+1} - E'$ that contains two k-links can be obtained.

Then we assume $|E_1| > 0$. Suppose that $(u, v) \in E_1$ is a $(k-1)$-link. We have $(u, u^{(k)}), (v, v^{(k)}), (u^{(k)}, v^{(k)}) \notin E'$. When $\delta(CQ_k^1 - (S_1 \cup E_1)) = 1$, we have $(x, x^{(k-1)}) \in S_1$, where x is the node of degree one in $CQ_k^1 - (S_1 \cup E_1)$. By the induction hypothesis, there are cycles of lengths ranging from 4 to 2^k in $CQ_k^1 - (S_1 \cup E_1 - \{(x, x^{(k-1)}), (u, v)\})$ that each contain at least two $(k-1)$-links. Let C denote any of these cycles. If (u, v) is contained in C, then a $(|C|+2)$-cycle in $CQ_{k+1} - E'$ that contains two k-links can be obtained (replacing (s, t) with (u, v)).

If (u, v) is not contained in C, then there exists a $(k-1)$-link $(w, w^{(k-1)}) \neq (x, x^{(k-1)})$ in C. A $(|C|+2)$-cycle in $CQ_{k+1} - E'$ that contains two k-links can be obtained (replacing (s, t) with $(w, w^{(k-1)})$). When $\delta(CQ_k^1 - (S_1 \cup E_1)) \ge 2$, arbitrarily select a $(k-1)$-link in S_1, and then l-cycles for $6 \le l \le 2^k + 2$ in $CQ_{k+1} - E'$ that each contain two k-links can be obtained similarly.

On the other hand, suppose that there is no $(k-1)$-link in E_1. Recall that each link $(s, t) \in S_0$ induces at least one link in $E_1 \cup E_c$. Since the links induced by (s, t) are $(k-1)$-links or k-links, we have $|S_0| \le |E_c|$, which implies $|S_0| + |E_0| \le |E_c| + |E_0| \le 2k-4$. Arbitrarily select a $(k-1)$-link, say (u, v), from E_0. By the induction hypothesis, there are cycles of lengths ranging from 4 to 2^k in $CQ_k^0 - (S_0 \cup E_0 - \{(u, v)\})$ that each contain at least two $(k-1)$-links. Let C denote any of these cycles. If (u, v) is contained in C, then a $(|C|+2)$-cycle in $CQ_{k+1} - E'$ that contains two k-links can be constructed. If (u, v) is not contained in C, then let $(w, w^{(k-1)})$ be a $(k-1)$-link of C. A $(|C|+2)$-cycle in $CQ_{k+1} - E'$ that contains two k-links can be obtained (replacing (u, v) with $(w, w^{(k-1)})$).

Case 2. $2^k + 3 \le l < 2^{k+1}$. Since $\delta(CQ_{k+1} - E') \ge 2$, we have $\delta(CQ_k^0 - E_0) \ge 1$. Three subcases: $|E_0| = 2k-3$, $|E_0| = 2k-4$ and $|E_0| \le 2k-5$, are discussed below.

Case 2.1. $|E_0| = 2k-3$. We have $|E_1| = |E_c| = 0$. There are at most two nodes of degree one in $CQ_k^0 - E_0$, for otherwise $|E_0| \ge 2(k-2) + (k-3) + 2 = 3k-5$ (there is no 3-cycle in CQ_k^0), which is a contradiction. Two node-disjoint links, say (u, v) and (x, y), are selected from E_0 so that they are incident with the nodes of degree one, if such nodes exist in $CQ_k^0 - E_0$. Clearly, $\delta(CQ_k^0 - (E_0 - \{(u, v), (x, y)\})) \geq 2$. By Lemma 3, there is a Hamiltonian cycle, denoted by C, in $CQ_k^0 - (E_0 - \{(u, v), (x, y)\})$.

If neither of (u, v) and (x, y) is contained in C, then obtain paths of lengths ranging from 2 to $2^k - 2$ from C. Let $P_{s,t}$ denote any of these paths, and $|P_{s,t}|$ be the length of $P_{s,t}$. By Lemma 2, there is a Hamiltonian path between $s^{(k)}$ and $t^{(k)}$ in $CQ_k^1 - E_1$. A $(|P_{s,t}| + 2^k + 1)$-cycle in $CQ_{k+1} - E'$ that contains two k-links can be constructed. If exactly one (assuming (u, v)) of (u, v) and (x, y) is contained in C, then obtain $P_{s,t}$ without containing (u, v), and a $(|P_{s,t}| + 2^k + 1)$-cycle in $CQ_{k+1} - E'$ that contains two k-links can be obtained all the same.

If both (u, v) and (x, y) are contained in C, then without loss of generality, assume that u, v, x, y appear clockwise in C. We use $P_{v,x}$ ($P_{y,u}$) to denote the path between v and x (between y and u) in C that does not contain (u, v) ((x, y)). Notice that $|P_{v,x}| + |P_{y,u}| = 2^k - 2$. We first construct cycles of lengths ranging from $2^k + 3$ to $|P_{v,x}| + 2^k + 1$ in $CQ_{k+1} - E'$ as follows. Obtain subpaths, denoted by $P_{s,t}$, of $P_{v,x}$ whose lengths range from 2 to $|P_{v,x}|$. By Lemma 2, there is a Hamiltonian path between $s^{(k)}$ and $t^{(k)}$ in $CQ_k^1 - E_1$. A $(|P_{s,t}| + 2^k + 1)$-cycle in $CQ_{k+1} - E'$ that contains two k-links can be obtained.

Then we construct cycles of lengths ranging from $|P_{v,x}| + 2^k + 2$ to $2^{k+1} - 1$ in $CQ_{k+1} - E'$ as follows. For each $|P_{v,x}| + 2^k + 2 \leq l \leq 2^{k+1} - 1$, obtain a subpath, denoted by $P_{s,t}$, of $P_{v,x}$ and a subpath, denoted by $P_{g,h}$, of $P_{y,u}$ so that $|P_{s,t}| + |P_{g,h}| = l - 2^k - 2$. By Lemma 5, there are two paths $P_{s^{(k)},g^{(k)}}$ and $P_{t^{(k)},h^{(k)}}$ in CQ_k^1 satisfying $V(P_{s^{(k)},g^{(k)}}) \cap V(P_{t^{(k)},h^{(k)}}) = \varnothing$ and $V(P_{s^{(k)},g^{(k)}}) \cup V(P_{t^{(k)},h^{(k)}}) = V(CQ_k^1)$. An l-cycle that contains four k-links of $CQ_{k+1} - E'$ can be constructed.

Case 2.2. $|E_0| = 2k - 4$. A $(|P_{s,t}| + 2^k + 1)$-cycle in $CQ_{k+1} - E'$ that contains two k-links can be obtained similarly to Case 2.1.

Case 2.3. $|E_0| \leq 2k - 5$. Similarly, there is at most one node of degree one in $CQ_k^0 - E_0$. Two subcases: $|E_1| \leq k - 3$ and $|E_1| = k - 2$, are discussed below.

Case 2.3.1. $|E_1| \leq k - 3$. When $\delta(CQ_k^0 - E_0) \geq 2$, by Lemma 3, there is a Hamiltonian cycle in $CQ_k^0 - E_0$. There are paths, denoted by $P_{s,t}$, of lengths ranging from 2 to $2^k - 2$ in the cycle so that $(s, s^{(k)}) \notin E_c$ and $(t, t^{(k)}) \notin E_c$. By Lemma 2, there is a Hamiltonian path between $s^{(k)}$ and $t^{(k)}$ in $CQ_k^1 - E_1$. A $(|P_{s,t}| + 2^k + 1)$-cycle in $CQ_{k+1} - E'$ that contains two k-links can be obtained.

When $\delta(CQ_k^0 - E_0) = 1$, we have $|E_0| \geq k - 1$, which further implies $|E_c| \leq k - 2$. Suppose that u is the node of degree one in $CQ_k^0 - E_0$. Arbitrarily select $(u, v) \in E_0$ such that $(v, v^{(k)}) \notin E_c$. Since $\delta(CQ_{k+1} - E') \geq 2$, we have $(u, u^{(k)}) \notin E_c$. We first construct cycles of lengths ranging from $2^k + 3$ to $2^k + 2^{k-1}$ as follows. By Lemma 3, there is a Hamiltonian cycle in $CQ_k^0 - (E_0 - \{(u, v)\})$. Besides, (u, v) is contained in the cycle. Let $P_{s,t}$ be a path in the cycle such that $P_{s,t}$ does not contain (u, v) and $(s, s^{(k)}), (t, t^{(k)}) \notin E_c$. There are $2^k - |P_{s,t}|$ choices for $P_{s,t}$, and we consider $2 \leq |P_{s,t}| \leq 2^{k-1} - 1$ (so, $2^k - |P_{s,t}| > |E_c|$). By Lemma 2, there is a Hamiltonian path between $s^{(k)}$ and $t^{(k)}$ in $CQ_k^1 - E_1$. A $(|P_{s,t}| + 2^k + 1)$-cycle in $CQ_{k+1} - E'$ that contains two k-links can be obtained.

Then we construct cycles of lengths ranging from $2^k + 2^{k-1} + 1$ to $2^{k+1} - 1$. By the induction hypothesis, there are cycles of lengths ranging from $2^{k-1} + 1$ to $2^k - 1$ in $CQ_k^0 - (E_0 - \{(u, v)\})$. Let C denote any of these cycles. If (u, v) is contained in C, then by Lemma 2, there is a Hamiltonian path between $u^{(k)}$ and $v^{(k)}$ in $CQ_k^1 - E_1$. Otherwise, select a link, say (s, t), of C such that $(s, s^{(k)}), (t, t^{(k)}) \notin E_c$. A $(|C| + 2^k)$-cycle in $CQ_{k+1} - E'$ that contains two k-links can be constructed (replacing (u, v) with (s, t) if (u, v) is not contained in C).

Case 2.3.2. $|E_1| = k - 2$. We have $|E_0| = k - 2$ or $k - 1$, and $|E_c| \leq 1$. A cycle of length $2^k + 3$ in $CQ_{k+1} - E'$ can be obtained as follows. Let (s, t) be a $(k - 1)$-link of a crossed 4-cycle in $CQ_k^0 - E_0$ such that $(s, s^{(k)}), (t, t^{(k)}) \notin E_c$. By Lemma 6, $(s^{(k)}, t^{(k)})$ is a $(k - 1)$-link of CQ_k^1 (and CQ_{k+1}), and by Lemma 12, there is a $(2^k - 1)$-cycle in $CQ_k^1 - (E_1 - \{(s^{(k)}, t^{(k)})\})$ that contains $(s^{(k)}, t^{(k)})$. A $(2^k + 3)$-cycle in $CQ_{k+1} - E'$ that contains two k-links can be constructed.

Cycles of lengths ranging from $2^k + 4$ to $2^{k+1} - 1$ can be obtained as follows. When $\delta(CQ_k^0 - E_0) \geq 2$, by the induction hypothesis, there are cycles of lengths ranging from 4 to $2^k - 1$ in $CQ_k^0 - E_0$ that each contain at least two $(k - 1)$-links. Let C denote any of these cycles, and (s, t) be a $(k - 1)$-link of C such that $(s, s^{(k)})$,

$(t, t^{(k)}) \notin E_c$. By Lemma 6, $(s^{(k)}, t^{(k)})$ is a $(k-1)$-link of CQ_k^1 (and CQ_{k+1}), and by Lemma 8, there is a Hamiltonian cycle in $CQ_k^1 - \{E_1 - (s^{(k)}, t^{(k)})\}$ that contains $(s^{(k)}, t^{(k)})$. A $(|C| + 2^k)$-cycle in $CQ_{k+1} - E'$ that contains two k-links can be obtained (replacing (u, v) with (s, t)).

When $\delta(CQ_k^0 - E_0) = 1$, assume that u is the node of degree one in $CQ_k^0 - E_0$. Arbitrarily select a d-link $(u, v) \in E_0$ such that d is odd and $(v, v^{(k)}) \notin E_c$. By Lemma 6, $(u^{(k)}, v^{(k)})$ is a d-link of CQ_k^1 (and CQ_{k+1}). By the induction hypothesis, there are cycles of lengths ranging from 4 to $2^k - 1$ in $CQ_k^0 - (E_0 - \{(u, v)\})$ that each contain at least two $(k-1)$-links. Let C denote any of these cycles. If (u, v) is contained in C, then by Lemma 8, there is a Hamiltonian cycle in $CQ_k^1 - \{E_1 - (u^{(k)}, v^{(k)})\}$ that contains $(u^{(k)}, v^{(k)})$. Otherwise, let (s, t) be a $(k-1)$-link of C such that $(s, s^{(k)}), (t, t^{(k)}) \notin E_c$. A $(|C| + 2^k)$-cycle in $CQ_{k+1} - E'$ that contains two k-links can be obtained (replacing (u, v) with (s, t) if (u, v) is not contained in C). ☐

4: Discussion and Conclusion

The *pancycle* problem on a graph G is to determine whether or not G contains cycles of lengths ranging from three to $|V(G)|$, and construct them if they exist. If all these cycles exist in G, then G is called *pancyclic*. Under the random fault model, the pancycle problem on the crossed cube was solved in [15], in which CQ_n could tolerate up to $n-2$ random link faults. In this paper, the conditional fault model was considered, and with the assumption that at least two fault-free links were incident with each node, the pancycle problem on the crossed cube was solved, in which CQ_n could tolerate up to $2n-5$ link faults.

The result is optimal, because there are distributions of $2n-4$ link faults in CQ_n that can prevent a fault-free Hamiltonian cycle in the faulty CQ_n. For example, let $u = 0^n$ (n consecutive 0's) and $v = 0^{n-2}1^2$, which are two nonadjacent nodes of CQ_n. If u and v have their 0-links and 1-links fault-free and the other d-links ($2 \le d \le n-1$) faulty, then the four fault-free links incident with u and v form a cycle of length four, as shown in Figure 7. When $n \ge 3$, there is no fault-free Hamiltonian cycle in the faulty CQ_n. It was shown in [10] that the probability that there are at least two fault-free links incident with each node of CQ_n is very close to 1.

REFERENCES

[1] Y. A. Ashir and I. A. Stewart, "Fault-tolerant embedding of Hamiltonian circuits in k- ary n-cube," *SIAM Journal on Discrete Mathematics*, vol. 15, no. 3, pp. 317-328, 2002.

[2] M. Y. Chan and S. J. Lee, "On the existence of Hamiltonian circuits in faulty hypercubes," *SIAM Journal on Discrete Mathematics*, vol. 4, no. 4, pp. 511-527, Nov. 1991.

[3] C. P. Chang, T. Y. Sung, and L. H. Hsu, "Edge congestion and topological properties of crossed cube," *IEEE Transactions on Parallel and Distributed Systems*, vol. 11, no. 1, pp. 64-80, Jan. 2000.

[4] K. Efe, "A variation on the hypercube with lower diameter," *IEEE Transactions on Computers*, vol. 40, no. 11, pp. 1312-1316, Nov. 1991.

[5] K. Efe, "The crossed cube architecture for parallel computation," *IEEE Transactions on Parallel and Distributed Systems*, vol. 3, no. 5, pp. 513-524, Sep. 1992.

[6] K. Efe, P. K. Blackwell, W. Slough, and T. Shiau, "Topological properties of the crossed cube architecture," *Parallel Computing*, vol. 20, pp. 1763-1775, 1994.

[7] J. Fan, X. Lin, X. Jia, "Node-pancyclicity and edge-pancyclicity of crossed cube," *Information Processing Letters*, vol. 93, pp. 133-138, 2005.

[8] F. Harary, "Conditional connectivity," *Networks*, vol. 13, pp. 347-357, 1983.

[9] W. T. Huang, Y. C. Chuang, L. H. Hsu, and J. M. Tan, "On the fault-tolerant hamiltonicity of crossed cubes," *IEICE Transactions on Fundamentals*, vol. E85-A, no. 6, pp. 1359-1371, Jun. 2002.

[10] H. S. Hung, "Fault-tolerant computation on crossed cubes with conditional faults," Ph.D. thesis, Department of Computer Science and Information Engineering, National Taiwan University, Taipei, Taiwan, 2006.

[11] P. Kulasinghe and S. Betayeb, "Embedding binary trees into crossed cubes," *IEEE Transactions on Computers*, vol. 44, no. 7, pp. 923-929, Jul. 1995.

[12] P. Kulasinghe, "Connectivity of the crossed cube," *Information Processing Letters*, vol. 61, pp. 221-226, 1997.

[13] C. H. Tsai, "Linear array and ring embeddings in conditional faulty hypercubes," *Theoretical Computer Science*, vol. 314, pp. 431-443, 2004.

[14] A. Y. Wu, "Embedding of tree networks into hypercubes," *Journal of Parallel and Distributed Computing*, vol. 2, no. 3, pp. 23-249, Aug. 1985.

[15] M. C. Yang, T. K. Li, J. M. Tan, and L. H. Hsu, "Fault-tolerant cycle-embedding of crossed cubes," *Information Processing Letters*, vol. 88, pp. 149-154, 2003.