Local Diagnosability under the PMC Diagnosis Model with Application to Star Graphs

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ABSTRACT

In this paper, we introduce a new measure of diagnosability, called local diagnosability, and provide a substructure for determining if a node is locally t-diagnosable under the PMC model. We prove that the local diagnosability of the star graph at each node is equal to its degree. Then, we propose a concept called strong local diagnosability property. We show that an n-dimensional star graph S_n has this strong property, $n\geq3$. Next, we prove that S_n keeps this strong property even if it has up to n-3 faulty edges. Besides, we propose a new diagnosis algorithm for general systems. The time complexity of our algorithm to diagnose all the faulty processors is bounded by O(N logN), where N is the total number of processors.

1: Introduction

In a multiprocessor system, processor fault identification plays an important role for reliable computing. The diagnosability of many interconnection networks have been widely studied [2], [3], [5], [11], [12]. The original diagnostic model was established by Preparata et al. [11]. This model, called PMC diagnosis model, has been extensively studied [6], [7], [8], [11]. In this model, a processor tests a subset of processors, then outputs the testing results. By analyzing the collection of all testing results, all of the faulty processors are identified.

The star graph [1] is a popular network topology for multiprocessor systems. An n-dimensional star graph is denoted by S_n, and the diagnosability of S_n is shown to be $n-1$ under the PMC model, $n \geq 3$ [9]. We observe that most previous literatures focused on the diagnosability of a system in a global sense, but ignored some local information. A system is t-diagnosable if all the faulty nodes in the system can be identified provided that the number of faulty nodes does not exceed t. However, it is possible to correctly point out all the faulty nodes in a t-diagnosable system when the number of faulty nodes is greater than t. For example, consider two systems which are m-diagnosable and n-diagnosable, respectively, and $m>>n$. The diagnosability of a new system which is generated by integrating these two systems with few edges in some way is upper bound by n. However, it is possible to point out all the faulty nodes even if the number of the faulty ones is between m and n. Hence, if only considering the global faulty/fault-free status, we lose some local details of the system.

In this paper, we propose a new measure of diagnosability, called local diagnosability, and study the local diagnosability of each processor instead of the whole system. We show that the diagnosability of a system can be determined by computing the local diagnosability of each node. We also provide a useful substructure to determine the local diagnosability of a processor under the PMC model. Based on the substructure, the local diagnosability of each node in a star graph is shown to be equal to its own degree. Consequently, we propose a concept called strong local diagnosability property. A system is said to have a strong local diagnosability property, if the local diagnosability of each node is equal to its degree. We show that S_n has this strong property. Then, we study the local diagnosability of an incomplete star graph. We show that S_n keeps this strong property even if it has up to ($n-3$) faulty edges. Finally, we propose a new diagnosis algorithm for general systems. The time complexity of our algorithm to diagnose a given processor is bounded by $O(\log N)$ and to diagnose all the faulty processors is bounded by $O(N \log N)$, where N is the total number of processors.

2: Preliminaries and previous results

For the graph definition and notation we follow [13]. A multiprocessor system can be modeled as a graph $G(V,E)$, where the set of nodes V represents processors and the set of edges E represents communication links between processors.

Let $G(V,E)$ be a graph and $v\in V(G)$ be a node. The degree of v is denoted by $deg_G(v)$ or simply $deg(v)$. The neighborhood $N_G(v)$ of v is the set of all nodes that are adjacent to v in G. Let G_1 be a subgraph of G, we shall write the node set of G_1 as $V(G_1)$. The following is a useful characterization for the distinguishability of two sets of nodes under the PMC model. The symmetric difference of two sets F_1 and F_2 is defined as the set $F_1\Delta F_2 = (F_1-F_2)\cup(F_2-F_1)$.

The PMC model[11] is presented by Preparata, Metze and Chien. In this model, a self-diagnosable system is represented by a directed graph $T(V,E)$ in which an edge directed from node u to node v means that u (the tester) can test v (the tested node). The outcome of a test (u,v) is 1(respectively, 0) if u evaluates v as faulty (respectively, fault-free). We assume that the testing results of fault-free nodes are always reliable and the testing results of faulty nodes are unreliable. The set of all testing results is called a syndrome. Formally, a syndrome is any function $\sigma : E \rightarrow \{0, 1\}$. The set of all faulty processors in the system is called a faulty set. This can be any subset of $V(T)$. For a given syndrome σ, a subset of nodes $F \subset V(T)$ is consistent with σ if the syndrome σ can be produced from the situation that all nodes in F are faulty and all nodes in V-F are fault-free. A syndrome σ is said to be consistent with a faulty set $F \subset V(T)$ if, for a $(u,v) \in E(T)$, such that $u \in V\text{-}F$, $\sigma(u,v) = 1$ if and only if $v \in F$. Since faulty testers can give arbitrary testing results, any syndrome consistent with a faulty set F can occur when faulty processors in the system are exactly those in F. The maximum number of faulty nodes that the system G can guarantee to identify is called the diagnosability of G, written as $t(G)$. Let σ_F be the set of all syndromes which could be produced if F is the set of faulty nodes. Two distinct sets $F_1, F_2 \subset V(G)$ are said to be distinguishable if $\sigma_{F1} \cap \sigma_{F2} = \varnothing$; otherwise, F_1, F_2 are said to be indistinguishable. We say (F_1, F_2) is a distinguishable pair if $\sigma_{F1} \cap \sigma_{F2} = \varnothing$; otherwise, (F_1, F_2) is an indistinguishable pair. We need some previous results concerning the t-diagnosable systems.

Lemma 1. [4] $G(V,E)$ is t-diagnosable if and only if, for any two distinct sets $F_1, F_2 \subset V$ with $|F_1| \leq t$ and $|F_2| \leq t$, (F_1, F_2) is a distinguishable pair.

Lemma 2. [4] For any two distinct sets $F_1, F_2 \subset V$, (F_1,F_2) is a distinguishable pair if and only if there exists a node $u \in V\text{-}(F_1 \cup F_2)$ and a node $v \in F_1 \Delta F_2$ such that $(u, v) \in E$

3: Local diagnosability

We observe that the traditional diagnosability discussed in most literatures describes the global status of a system. For this reason, we are motivated to study the local status of each processor instead of the whole system. Given a single node, we require only identifying the status of this particular node correctly. We now propose the following concept.

Definition 1. G is locally t-diagnosable at node v if, given a syndrome σ_F produced by a set of faulty nodes $F \subset V$ containing node v with $|F| \leq t$, every set of faulty nodes F' consistent with σ_F and $|F'| \leq t$, must also contain node v.

Definition 2. The local diagnosability of node v, written as $t_l(v)$, is defined to be the maximum value of t such that G is locally t-diagnosable at node v.

The following result is another point of view for checking if a node is locally t-diagnosable.

Lemma 3. G is locally t-diagnosable at node v if and only if, for any two distinct sets of nodes $F_1, F_2 \subset V$, $|F_1| \leq t$, $|F_2| \leq t$ and $v \in F_1 \Delta F_2$, (F_1, F_2) is a distinguishable pair.

In the following, we study some properties of a node being locally t-diagnosable, and its relationship between a system being t-diagnosable.

Proposition 1. Let $G(V,E)$ be a graph and $v \in V$ be a node with $deg(v) = n$. The local diagnosability of node v is at most n.

Proposition 2. Let $G(V,E)$ be a graph. G is t-diagnosable if and only if G is locally t-diagnosable at every node.

By Definition 2 and Proposition 2, we can prove that the diagnosability of a system is equal to the minimum local diagnosability of all nodes of the system. Thus, we have the following theorem.

Theorem 1. The diagnosability of G is t if and only if

$$min\{t_l(v) \mid \text{for every } v \in V\} = t.$$

From Theorem 1, we can identify the diagnosability of a system by computing the local diagnosability of each node. Because many well-known systems are node-symmetric, the diagnosability of these system can be easily identified by this effective method.

Before studying the local diagnosability of a node, we need some definitions for further discussion. Let S be a set of nodes and v be a node not in S. After deleting the nodes in S from G, we use C_v to denote the connected component which node v belongs to. Now, we propose a necessary and sufficient condition for verifying if a system is locally t-diagnosable at a given node v.

Theorem 2. G is locally t-diagnosable at node v if and only if, for each set of nodes $S \subset V$ with $|S| = p$, $0 \leq p \leq t\text{-}1$ and $v \notin S$, the connected component, which v belongs to in G - S, has at least $2(t - p)+1$ nodes.

Proof. To prove the necessity, we assume that G is locally t-diagnosable at node v. If the result does not hold, there exists a set of nodes $S \subset V$ with $|S| = p$, $0 \leq p \leq t - 1$, $v \notin S$ such that the connected component C_v has strictly less than $2(t - p)+1$ nodes, $|V(C_v)| \leq 2(t - p)$. We then arbitrarily partition $V(C_v)$ into two disjoint subsets, $V(C_v)=S_1 \cup S_2$ with $|S_1| \leq t - p$, $|S_2| \leq t - p$. Let $F_1=S_1 \cup S$ and $F_2=S_2 \cup S$. It is clear that $|F_1| \leq (t\text{-}p)+p=t$, $|F_2| \leq (t\text{-}p)+p=t$, the node $v \in F_1 \Delta F_2$ and there is no edge between $V\text{-}(F_1 \cup F_2)$ and $F_1 \Delta F_2$. By Lemma 3, (F_1, F_2) is an indistinguishable pair. This contradicts the assumption that G is locally t-diagnosable at node v.

We now prove the sufficiency by contradiction. Suppose G is not locally t-diagnosable at node v, then, there exists an indistinguishable pair (F_1, F_2) with $|F_1| \leq t$,

$|F_2| \le t$ and $v \in F_1 \Delta F_2$. By Lemma 2, there is no edge between V-$(F_1 \cup F_2)$ and $F_1 \Delta F_2$. Let $S=F_1 \cap F_2$ with $|S|=p$, $0 \le p \le t - 1$ and $v \notin S$. $F_1 \Delta F_2$ is disconnected from other parts after removing all the nodes in S from G. We observe that $|F_1 \Delta F_2| \le 2(t - p)$. Thus, the connected component C_v has at most $2(t - p)$ nodes and $|V(C_v)| \le 2(t - p)$. This contradicts the assumption that the connected component C_v has to satisfy $|V(C_v)| \le 2(t - p) + 1$. Hence, the theorem holds. □

We now propose a special substructure. This provides us with an efficient and simple method to identify the local diagnosability of each node of a system under the PMC diagnosis model.

Definition 3. Let $G(V,E)$ be a graph, $v \in V$ be a node and k be an integer, $k \ge 1$, a substructure $H(v, k)$ of order k at node v is defined to be the following graph,

$$H(v, k) = [V(v; k), E(v; k)]$$

which is composed of $2k$+1 nodes and of $2k$ edges as illustrated in Figure 1, where

- $V(v; k) = \{v\} \cup \{x_i, y_i \mid 1 \le i \le k\}$,
- $E(v; k) = \{(v, x_i), (x_i, y_i) \mid 1 \le i \le k\}$.

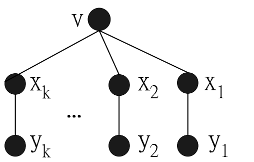

Figure 1: A substructure $H(v; k)$ of order k at node v.

Following Theorem 2 and Definition 3, we propose a sufficient condition for verifying if it is locally t-diagnosable at a given processor in a system.

Theorem 3. G is locally t-diagnosable at node v if G contains a substructure $H(v; t)$ of order t at node v.

Proof. We use Theorem 2 to prove this result. Assume that G contains a substructure $H(v; t)$ at node v. Let $e_i=(x_i, y_i)$ be the edge for each i, $1 \le i \le t$, with respect to $H(v; t)$. The number of nodes of the connected component including node v is at least $2t + 1$. Let $S \subset V(G)$ be a set of nodes with $|S| = p$, $0 \le p \le t - 1$ and $v \notin S$. After deleting S from $V(G)$, there are at least $(t - p)$ complete e_i's still remain in $T_1(v;t)$. Therefore, the number of nodes of the connected component C_v is at least $2(t-p)+1$. By Theorem 2, G is locally t-diagnosable at node v. The proof is complete. □

By Theorem 3, we have the following result.

Theorem 4. Let $G(V,E)$ be a graph and $v \in V$ be a node with $deg(v) = n$. The local diagnosability of node v is n if G contains a substructure $H(v; n)$ of order n at node v.

4: Strong Local Diagnosability Property

We use the star graph as an example to introduce our concept of the strong local diagnosability property. An n-dimensional star graph S_n is an undirected graph consisting of n! nodes and $(n$-1$)n!/2$ edges. The set of nodes $V(S_n)=\{u_1u_2\ldots u_n \mid u_i \in \langle n \rangle$ and $u_i \ne u_j$ for $i \ne j\}$, where $\langle n \rangle$ is the set $\{1, 2, \ldots, n\}$. The adjacent is defined as follows: $u_1u_2\ldots u_i\ldots u_n$ is adjacency to $v_1v_2\ldots v_i\ldots v_n$ through an edge of dimension i with $2 \le i \le n$ if $v_j = u_j$ for $j \notin \{1, i\}$ $v_i = u_i$ and $v_i = u_1$. For example, in S_4 containing 4! nodes, two nodes 1234 and 4231 are neighbors and joined through an edge labeled 4. In S_n, each node is connected to n-1 neighbors by n-1 edges. Each S_n can be decomposed into n subgraph S_{n-1}. The star graph S_2, S_3 and S_4 are shown in Figure 2. Now we propose the following two concepts.

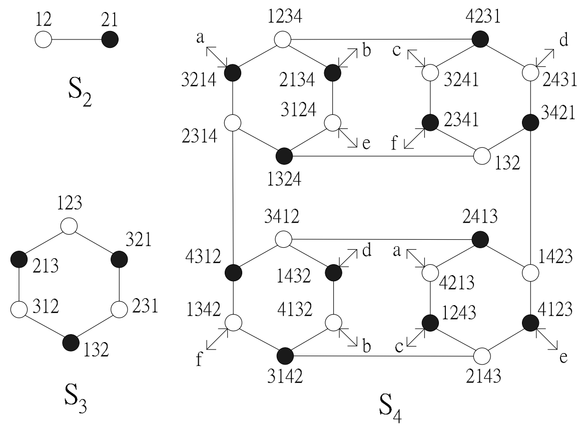

Figure 2: The star graph S_2, S_3 and S_4.

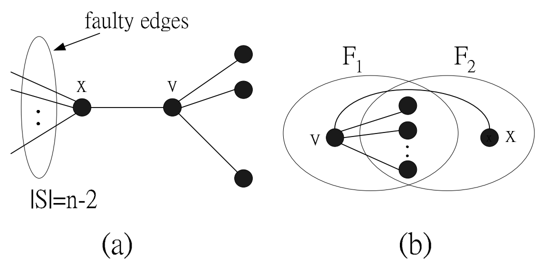

Figure 3: An indistinguishable pair (F_1, F_2), where $|F_1| = |F_2| = n - 1$.

Definition 4. A node v has the strong local diagnosability property if the local diagnosability of node v is equal to its degree.

Definition 5. A graph has the strong local diagnosability property if, every node in the graph has the strong local diagnosability property.

Following Definition 4 and 5, and by using a simple induction, we can prove the following theorem. Due to the page limit, we omit this routine proof.

Theorem 5. S_n has the strong local diagnosability property, $n \ge 3$.

We now consider a system which is not node-symmetric. Let $G(V,E)$ be a graph and $S \subset E(G)$ be a set of edges. Removing the edges in S from G, the degree of each node in the resulting graph G - S is called the remaining degree of v, and is denoted by $deg_{G-S}(v)$. We consider a faulty star graph S_n with a faulty set $S \subset E(S_n)$, $n \geq 3$. We shall prove that S_n has the strong local diagnosability property even if it has up to $(n - 3)$ faulty edges. The number $(n - 3)$ is optimal in the sense that a faulty S_n cannot be guaranteed to have this strong property if there are $(n\text{-}2)$ faulty edges. As shown in Figure 3, we take a node $v \in V(S_n)$ and a node x which is an adjacent neighbor of v. Let $S = \{(y, x) \in E(S_n) \mid$ node y is directly adjacent to $x\} - \{(v, x)\}$, then $|S| = n-2$ and the remaining degree of v in S_n-S is n-1. Let $F_1 = (N_{Sn-S}(v) - \{x\}) \cup \{v\}$ and $F_2 = N_{Sn-S}(v)$, then $|F_1| = |F_2| = n - 1$ and $v \in F_1 \Delta F_2$. It is clear that there is no edge between V-$(F_1 \cup F_2)$ and $F_1 \Delta F_2$. By Lemma 2, (F_1, F_2) is an indistinguishable pair, hence $t_l(v) \neq deg_{Sn-S}(v) = n$-1. Therefore, S_n-S may not have this strong property, if $|S| \geq n - 2$.

Again, we can use induction method to prove the following theorem.

Theorem 6. Let S_n be an n-dimensional star graph with $n \geq 3$, and $S \subset E(S_n)$ be a set of edges, $0 \leq |S| \leq n - 3$. Removing all the edges in S from S_n, the local diagnosability of each node is still equal to its remaining degree.

We have the following corollary.

Corollary 1. Let S_n be an n-dimensional star graph with $n \geq 3$, and $S \subset E(S_n)$ be a set of edges, $0 \leq |S| \leq n - 3$. Then, S_n - S has the strong local diagnosability property.

5: A Diagnosis Algorithm

We now introduce a diagnosis algorithm to determine if a node is faulty or not for a given syndrome under the PMC model. Given a substructure $H(v; n)$ of order n at node v, there are communication links between v and x_i, x_i and y_i, for all $1 \leq i \leq n$, x_i and y_i can be the tester of the PMC model. After the test, each tester has a testing result denoted by 0 (1, respectively) representing the approval (disapproval, respectively). We define $r_i = (r^1, r^2)$, where r^1 is the result of x_i testing v and r^2 is the result of y_i testing x_i. Then, r_i can be in one of the four different states which are $r(0) = (0, 0)$, $r(1) = (0, 1)$, $r(2) = (1, 0)$ and $r(3) = (1, 1)$ (as illustrated in Figure 4). Let $R(\mathrm{k})$ be the collection of all $r(\mathrm{k})$, for all $0 \leq k \leq 3$. Obviously, $\sum_{k=0}^{3} |R(k)| = n$.

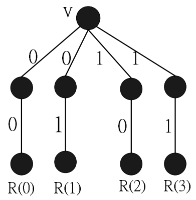

Figure 4: four different output states.

Suppose that there is a substructure $H(v; n)$ of order n at node v, where v has degree n. By Theorem 4, the local diagnosability of v is limited to n. Therefore, we may not be able to identify all the faulty nodes, if the number of faulty nodes in $H(v; n)$ is $n+1$ or more. Hence, we assume that the number of faulty nodes is at most n. Under this assumption, we propose the following algorithm to determine whether node v is faulty or not.

Theorem 7. Let v be a node with degree n in $G(V,E)$. Suppose that there is a substructure $H(v; n)$ of order n at node v and the number of faulty nodes is at most n. The following two conditions are satisfied:

1. the node v is fault-free if $|R(0)| \geq |R(2)|$, and
2. the node v is faulty if $|R(0)| < |R(2)|$.

Proof. Let $l_i = (x_i, y_i)$ be an ordered double, $1 \leq i \leq n$, with respect to $H(v; n)$. First, we prove the condition 1 by contradiction. Assume that v is faulty, then the counting of all the other faulty nodes is as follows:

For those l_i with result $r(0)$, there are at least 2 faulty nodes which are x_i, y_i.

For those l_i with result $r(1)$, there is at least 1 faulty node which is x_i.

For those l_i with result $r(2)$, the number of faulty nodes is uncertain.

For those l_i with result $r(3)$, there is at least 1 faulty node which is either x_i or y_i.

Thus, the number of faulty nodes is at least $1 + 2|R(0)| + |R(1)| + |R(3)| = \sum_{k=0}^{3} |R(\mathrm{k})| + (1 + |R(0)| - |R(2)|)$. By the assumption that $|R(0)| \geq |R(2)|$, the number of faulty nodes is strictly more than n. This contradicts to the assumption that the number of faulty nodes is at most n. Therefore, the node v is fault-free.

Next, we prove the condition 2 by contradiction again. Assume that v is fault-free, then the counting of all the other faulty nodes is as follows:

For those l_i with result $r(0)$, the number of faulty nodes is uncertain.

For those l_i with result $r(1)$, there is at least 1 faulty node which is either x_i or y_i.

For those l_i with result $r(2)$, there are at least 2 faulty nodes which are x_i and y_i.

For those l_i with result $r(3)$, there is at least 1 faulty node which is x_i.

Thus, the number of faulty nodes is at least $|R(1)| + 2|R(2)| + |R(3)| = \sum_{k=0}^{3} |R(k)| + (|R(2)| - |R(0)|)$. By the assumption that $|R(0)| < |R(2)|$, the number of faulty nodes is larger than n. This contradicts to the assumption that the number of faulty nodes is at most n. Therefore, the node v is faulty.

This completes the proof. □

We now measure the time complexity of our algorithm to diagnose all the faulty nodes in a system. For many well-know general systems with N nodes, the degree of each node is in the order of $\log N$. For example, the n-dimensional Hypercube Q_n has $N = 2^n$ nodes and the degree of each node is n, $n = \log N$; the n-dimensional star graph S_n has $N = n!$ nodes and the degree of each node is $n-1 = O(n) = O(\log N/\log n) = O(\log N/\log \log N)$. We assume that a testing result of each tester is directly stored in a syndrome table. Given a substructure $H(v; n)$ of order n at node v, assume the time for looking up the testing result of a tester in the syndrome table is constant c. Then, the time needed for determining the faulty or fault-free status of a node v is $2c\log N = O(\log N)$. Consequently, the total time to diagnose all the faulty nodes is bounded by $O(N \log N)$.

6: Conclusions

In this paper, we propose a new concept of local diagnosability for a system and derive a substructure for determining if a system is locally t-diagnosable at a given node. Through this concept, the diagnosability of a system can be determined by computing the local diagnosability of each node. We also introduce a concept for system diagnosis, called strong local diagnosability property. We prove that the star graph has this strong property. Then, we consider an n-dimensional faulty star graph S_n with a set of faulty edges $S \subset E(S_n)$, $0 \leq |S| \leq n-3$, $n \geq 3$. We prove that a faulty star graph S_n-S keeps this strong property. According to Theorem 1, the global diagnosability of S_n-S is equal to the minimum local diagnosability of all nodes. Finally, we propose a new diagnosis algorithm for general systems. The time complexity of our algorithm to diagnose all the faulty processors is bounded by $O(N \log N)$, where N is the total number of processors.

We use the star graph as an example to introduce the concepts of the local diagnosability, the local structure and the strong local diagnosability property. In fact, many well-known systems also have the local structure and this strong property. There are several different fault diagnosis models in the area of diagnosability. It is worth investigating, under various models, whether a system has this strong local diagnosability property after removing some edges.

In [10], Lai et al. introduced a measure of diagnosability called conditional diagnosability by restricting that a faulty set cannot contain all the neighbors of any node. Therefore, it is also an attractive work to develop more different measures of local diagnosability based on network reliability.

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