A Linear Time Algorithm for Solving the Incidence Coloring Problem of Chordal Graphs

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Abstract

An *incidence* of G consists of a vertex and one of its incident edge in G. The *incidence coloring problem* is a variation of vertex coloring problem. The problem is to find the minimum number (called *incidence coloring number*) of colors needed to dye every incidence of G so that the adjacent incidences do not dye the same color. A graph G is called a chordal (or triangulated) graph if and only if there is no induced cycle of length greater than 3 in G. In this paper, we propose a linear time algorithm for incidence-coloring a chordal graph. Further, we prove that the incidence coloring number of a chordal graph is $\Delta(G)+1$, where $\Delta(G)$ is the maximum degree of G.

Keywords: chordal graphs, incidence coloring problem, perfect elimination ordering.

1. Introduction

The *incidence set* of a graph $G=(V,E)$ is defined as $I(G)=\{(v,e): v\in V, e\in E, v$ is incident with $e\}$, where V and E are the vertex and edge, respectively, sets of G. Two incidences (v_1,e_1) and (v_2,e_2) are *adjacent* if one of the following conditions holds: (i) $v_1=v_2$, (ii) $e_1=e_2$, or (iii) the edge v_1v_2 equals to e_1 or e_2.

An *incidence coloring function* σ of G is a mapping from $I(G)$ to a *color set* such that adjacent incidences of G are assigned different colors. For example, $\sigma(v,e)=c$ means that the incidence (v,e) is colored with c. The *incidence coloring number* of G, denoted by $\chi_\iota(G)$, is the smallest size of the color set. The *incidence coloring problem* is to find the incidence coloring number of a given graph. In [3], Brualdi and Massey first defined the problem as a variation of vertex coloring problem.

Let $\Delta(G)$ be the maximum degree of a graph G. Then, it is obvious that $\chi_\iota(G)\geq\Delta(G)+1$ if G has at least one edge. Brualdi and Massey have proved that the incidence coloring number of a given graph G is at most $2\Delta(G)$ [3]. They also conjectured that any graph G can be incidence-colored with $\Delta(G)+2$ colors. However, their conjecture was disproved by Guiduli [7].

In [7], Guiduli also showed that the incidence coloring problem is a special case of *directed star arboricity* which was introduced by Algor and Alon [1]. Meanwhile, the directed star arboricity problem has application in the WDM (Wavelength Division Multiplexing) of a star optical network [2].

As for the incidence coloring number of special classes of graphs, the following results are well-known:

- For every $n\geq 2$, $\chi_\iota(K_n)=n=\Delta(K_n)+1$ [3], where K_n is a complete graph with n vertices.
- For every $m\geq n\geq 2$, $\chi_\iota(K_{m,n})=m+2=\Delta(K_{m,n})+2$ [3], where $K_{m,n}$ is a complete bipartite graph with m, n vertices in two partite sets.
- For every tree T of order $n\geq 2$, $\chi_\iota(T)=\Delta(T)+1$ [3].
- For every Halin graph G with $\Delta(G)\geq 5$, $\chi_\iota(G)=\Delta(G)+1$ [12].

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- For every outerplanar graph G with $\Delta(G) \geq 4$, $\chi_\iota(G) = \Delta(G) + 1$ [12].

In [11], Shiu et al. showed that Brualdi's conjecture holds for cubic Hamiltonian graphs and some other cubic graphs. In [9], Maydanskiy proved that $\chi_\iota(G) \leq 5$ for any graph with $\Delta(G) = 3$. In [8], Huang et al. showed that square mesh, hexagonal meshes and honeycomb meshes can be incidence-colored with $\Delta(G) + 1$ colors [8]. In [4], Dolama et al. proved that incidence coloring of every k-degenerated graph G is at most $\Delta(G) + 2k - 1$.

Chordal graphs form an important and widely studied subclass of perfect graphs. Further, chordal graphs have applications in many practical areas such as scheduling, Gaussian elimination on sparse matrices, and so on [6]. In this paper, we shall propose a linear time algorithm for incidence-coloring a chordal graph G. In addition, we also prove that the incidence coloring number of a chordal graph G is $\Delta(G) + 1$.

The remaining part of this paper is organized as follows. In Section 2, we introduce chordal graphs and some important properties of chordal graphs. Section 3 contains our incidence coloring algorithm and the correctness proof of the algorithm. The last section gives our conclusion and idea for future work.

2. Preliminaries

An undirected graph is *chordal* if and only if there is no induced cycle of length greater than three. Let $N(v)$ denote the set of neighbors of v and $N[v]$ denote the set of $\{v\} \bigcup N(v)$. A vertex v of a graph $G = (V, E)$ is called *simplicial* if $N(v)$ induces a clique in G. An *elimination ordering* ρ of a graph G is a bijection $\rho : \{1, 2, \ldots, n\} \to V$, where $n = |V|$. Accordingly, $\rho(i)$ is the i-th vertex in the elimination ordering and $\rho^{-1}(v)$, $v \in V$, gives the position of v in ρ. A *perfect elimination ordering* (PEO) is an elimination ordering $\rho = (v_1, v_2, \ldots, v_n)$, where v_i $(1 \leq i \leq n)$ is a simplicial vertex in the subgraph induced by vertex set $\{v_i, v_{i+1}, \ldots, v_n\}$. The following theorem is well-known.

Theorem 1 (Fulkerson and Gross [5]; Golumbic [6]) *An undirected graph is chordal if and only if it has a perfect elimination ordering.*

There exists many algorithms to generate PEOs for a chordal graph. For example, the lexicographic breadth-first search algorithm proposed by Rose et al. is the most famous one [10]. Given a PEO ρ of a chordal graph G, we have the following definitions.

A vertex $u \in N(v)$ is called a *higher neighbor* of v if $\rho^{-1}(u) > \rho^{-1}(v)$. The set of higher neighbors of v will be denoted by $N_h(v)$, i.e.,

$$N_h(v) = \{u \in N(v) : \rho^{-1}(u) > \rho^{-1}(v)\}.$$

Similarly, we define the set of *lower neighbors* of v and denote it by $N_l(v)$, i.e.,

$$N_l(v) = \{u \in N(v) : \rho^{-1}(u) < \rho^{-1}(v)\}.$$

In addition, let $d_h(v)$ and $d_l(v)$ denote the size of $N_h(v)$ and $N_l(v)$, repectively.

A chordal graph G can be constructed by reversing a PEO ρ. That is, starting with an empty graph, we add vertices according to the order $\rho(n), \rho(n-1), \ldots, \rho(1)$ and make each added vertex v adjacent to all vertices in $N_h(v)$. Let $G[v]$ be the subgraph induced by $\{v\} \bigcup N_h(v)$, or $N_h[v]$, in G. By Theorem 1, it turns out that $G[v]$ is a clique. We can determine the incidence coloring number of $G[v]$ by using a previous result proposed in [3].

Lemma 2 *For each vertex v in a chordal graph G, $\chi_\iota(G[v]) = d_h(v) + 1$.*

Proof. Brualdi and Massey have proved that for every $n \geq 2$, $\chi_\iota(K_n) = \Delta(K_n) + 1$. Since $G[v]$ is a complete subgraph induced by $N_h[v]$ in G, the incidence coloring number of $G[v]$ must be $d_h(v)+1$. □

For incidence-coloring a complete graph G, we assign distinct color to every vertex v in G, and call it the *attached color* of v. Then, the attached color of vertex v is used to dye incidence (u, uv) for each vertex $u \in N(v)$. This scheme can be extended to dye incidences of a chordal graph.

The *union* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $G_1 \bigcup G_2$, is the graph with vertex set $V_1 \bigcup V_2$ and edge set $E_1 \bigcup E_2$. We consider the union of two subgraphs of a chordal graph G. For $1 \leq i \leq n-1$, let S_i be the vertex set $\{\rho(n), \rho(n-1), \ldots, \rho(i)\}$ and $G[S_i]$ be the subgraph induced by S_i. Then, we have $G[S_i] = G[S_{i+1}] \bigcup G[\rho(i)]$.

Theorem 3 *Let $S_i = \{\rho(n), \rho(n-1), \ldots, \rho(i)\}$ be a vertex set corresponding to a PEO ρ of a chordal graph G. Then, $\chi_\iota(G[S_i]) = \Delta(G[S_i]) + 1$ for $1 \leq i \leq n-1$, where n is the number of vertices in G.*

Proof. We prove the lemma by induction on the cardinality of S_i. When $i = n-1$, $G[S_{n-1}]$ is a 2-clique that consists of vertices $\rho(n)$ and $\rho(n-1)$ if G is connected. It is obviously true that $\chi_\iota(G[S_{n-1}]) = \Delta(G[S_{n-1}]) + 1 = 2$ since two attached colors are required for a 2-clique. (In case that G is disconnected, we can get the incidence coloring number of individual connected component and solve the problem.)

Suppose $\chi_\iota(G[S_{k+1}]) = \Delta(G[S_{k+1}]) + 1$ is true. There are two conditions after $\rho(k)$ is added to the simplicial vertex set S_{k+1}. One condition is that every vertex in $G[\rho(k)]$ has the maximum degree and no other vertex in $G[S_k]$ has the maximum degree. In this case, we have $\Delta(G[S_k]) = \Delta(G[S_{k+1}]) + 1$ since the increased degree must due to the added $\rho(k)$. Based on the coloring scheme used in complete graph, incidence $(\rho(k), \rho(k)u)$ is dyed with the attached color of vertex u for every vertex $u \in N_h(\rho(k))$. As for the attached color of $\rho(k)$, it is inevitable to assign a new color. This newly-assigned color is used to dye incidence $(u, u\rho(k))$ for every vertex $u \in N_h(\rho(k))$. As a result, $\chi_\iota(G[S_k]) = \chi_\iota(G[S_{k+1}]) + 1 = \Delta(G[S_{k+1}]) + 1 = \Delta(G[S_k])$.

The other condition is that there exists a vertex $w \in G[S_{k+1}]$ and $w \notin G[\rho(k)]$ such that $d_h(\rho(k))$ is less than or equal to the degree of w. That is, $\Delta(G[S_{k+1}]) = \Delta(G[S_k])$. Since $w \notin N_h(\rho(k))$, The attached color of w can be assigned to the attached color of vertex $\rho(k)$ and complete the incidence-coloring work. In this case, $\chi_\iota(G[S_k]) = \chi_\iota(G[S_{k+1}]) = \Delta(G[S_{k+1}]) = \Delta(G[S_k])$. □

Furthermore, since $G[S_1] = G$, we have the following corollary.

Corollary 4 *For a chordal graph G, $\chi_\iota(G) = \Delta(G) + 1$.*

3. The Incidence Coloring Algorithm

In this section, we present a linear time algorithm for incidence coloring a chordal graph. At first, we determine a PEO of the chordal graph. Based on the reversed order of the PEO, we process each vertex and compute the incidence coloring number of the graph.

Let $IC[v_i]$ $(i = 1, \ldots, n)$ be an array that records the attached colors of corresponding vertices. All incidences adjacent to a common vertex v are colored with color $IC[v]$ in our algorithm.

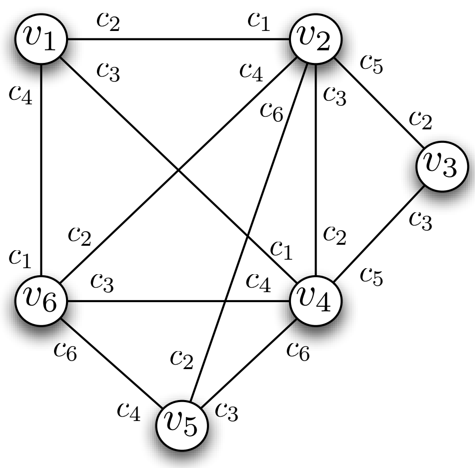
We show the algorithm for incidence-coloring a chordal graph as follows.

Algorithm InciColor_Chordal
Input: A chordal graph G.
Output: Incidence coloring number $\chi_\iota(G)$.
  Step 1. Find a PEO ρ in G.
  Step 2. Incidence coloring G.
    $k \leftarrow 0$;
    For $i = n$ downto 1 **do**
      If $IC[\rho(i)]$ is null **then**
        $k \leftarrow k + 1$;
        $IC[\rho(i)] \leftarrow c_k$;
      Endif
      For each $u \in N_l(\rho(i))$ **do**
        $\sigma(u, u\rho(i)) \leftarrow IC[\rho(i)]$;
        If $IC[u]$ is null **then**
          $k \leftarrow k + 1$;
          $IC[u] \leftarrow c_k$;
        Endif
        $\sigma(\rho(i), \rho(i)u) \leftarrow IC[u]$;
      Enddo
    Enddo
    $\chi_\iota(G) \leftarrow k$;
  Step 3. Output the incidence coloring number $\chi_\iota(G)$.
End of Algorithm InciColor_Chordal

We give an example to illustrate the incidence coloring algorithm. Considering the chordal graph G shown in Figure 1(a), a PEO $\rho = \{v_3, v_5, v_6, v_4, v_2, v_1\}$ of G is shown in Figure 1(b). We start with vertex v_1. Since $IC[v_1]$ is null, we assign a color c_1 to $IC[v_1]$. Three vertices v_2, v_4 and v_6 are adjacent to v_1, i.e., $N_l(v_1) = \{v_2, v_4, v_6\}$. Accordingly, we get $\sigma(v_2, v_2v_1) = \sigma(v_4, v_4v_1) = \sigma(v_6, v_6v_1) = c_1$. Then, we assign c_2, c_3 and c_4 to $IC[v_2]$, $IC[v_4]$ and $IC[v_6]$, respectively, such that $\sigma(v_1, v_1v_2) = c_2$, $\sigma(v_1, v_1v_4) = c_3$, and $\sigma(v_1, v_1v_6) = c_4$. As processing vertex v_2, since $IC[v_2]$ has been set to c_2 and $N_l(v_2) = \{v_3, v_4, v_5, v_6\}$, we get $\sigma(v_3, v_3v_2) = \sigma(v_4, v_4v_2) = \sigma(v_5, v_5v_2) = \sigma(v_6, v_6v_2) = c_2$. Then, we assign c_5 and c_6 to $IC[v_3]$ and $IC[v_5]$, respectively, and obtain $\sigma(v_2, v_2v_4) = c_3$, $\sigma(v_2, v_2v_6) = c_4$, $\sigma(v_2, v_2v_3) = c_5$, and $\sigma(v_2, v_2v_5) = c_6$. Subsequent vertices v_4, v_6, v_5 and v_3 are processed in the same manner. Finally, we obtain $\chi_\iota(G) = 6$.

Theorem 5 *Algorithm InciColor_Chordal can correctly incidence-color a chordal graph in $O(m + n)$ time, where m and n are the size and order of the graph, respectively.*

Proof. Algorithm InciColor_Chordal is correct since it is an implementation of the constructive proof of Theorem 3. The algorithm computes the incidence coloring number of a chordal graph $G(V, E)$ based on a PEO that can be obtained in $O(m + n)$ time. To color every incidence in the graph, it takes $\sum_{v \in V} 2d_l(v) = 2m$ time. Thus, the overall time requirement is $O(m + n)$. □



(a)

i	$\rho(i)$	$N_l(\rho(i))$	$IC[\rho(i)]$
6	v_1	$v_2\ v_4\ v_6$	c_1
5	v_2	$v_3\ v_4\ v_5\ v_6$	c_2
4	v_4	$v_3\ v_5\ v_6$	c_3
3	v_6	v_5	c_4
2	v_5	–	c_6
1	v_3	–	c_5

(b)

Figure 1: An example of Algorithm InciColor_Chordal: (a) a chordal graph and its incidence coloring; (b) a PEO of the graph and the related data of incidence coloring.

4. Concluding Remarks

We have proposed a linear time algorithm for incidence-coloring a chordal graph and proved that the incidence coloring number of a chordal graph G is $\Delta(G) + 1$. The future research works are summarized as two directions. One is to find out other classes of graphs which have the property of $\chi_\iota(G) = \Delta(G) + 1$. Another is to find out other variations of graph coloring problem which have solutions in complete graphs, and to extend the solutions to chordal graphs.

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