# Minimum Variance Control of a General Supply Chain Unit

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## Abstract

In a supply chain system, it is very important to forecast the need of the market and maintain a reasonable inventory level to satisfy the customer demand. On the other hand, it is also very important to keep the whole chain operated in a stable condition. However, these two aims may conflict to each other in many cases. This is due to large demand variability in a supply chain system, and that results in the so-called "bullwhip effect", the distortion of demand in upstream activities. In this study we present a predictive controller to solve this problem. In the dynamic supply chain system, the sequences of the customer demands, order amounts, and inventory level all belong to the numbers of time series. It is possible to translate the customer demand time series models into a general ARIMA model. On the other hand, the inventory level of a supply chain can be modeled using material and information balances. Based on the above two techniques, a minimum variance control (MVC) theory can be implemented to design the ordering law which can track the need of the market well and eliminate the bullwhip effect. To accomplish these two needs, a predictive controller is formulated, for the first time, for the supply chain system. The objective function is formulated to make it possible to tune the controller parameters to minimize the excess inventory and/or the backorder. It is shown that this model based control law can solve effectively this bullwhip phenomenon and manage a proper inventory level no matter the customer demand model is a stationary or non-stationary model.

**Keywords: supply chain, predictive model, minimum variance control, bullwhip effect.**

## 1. Introduction

The purpose of this work is to formulate a general predictive inventory level controller for a supply chain unit such that the system can maximize its profit from the prevention of large excess inventory and dissatisfaction of the customers.

A supply chain system is essentially a dynamic balance of material and information flow and ordering policy serves as an inventory control system. In the past, the design for the supply chain model is conventionally founded on continuous time domain. The Laplace transformation technique is naturally used to solve the supply chain system. Towill [1] presented a classical control concept and adopted the Lapace operator to analyze the performance of the inventory management. Perea-López et al.[2,3] proposed a dynamic model to demonstrate the behaviors of a supply chain system. Dejonckheere and his coworkers [4,5] offered an order replenishment rule, and then applied the transfer functions of z-transform-inverts to discuss the bullwhip effect phenomenon. Meanwhile, the authors, Lin et al.[6] proposed a discrete supply chain model to investigate its dynamics behaviors and used the cascade control to obtain better performances of the system.

A special phenomenon existed in place-order action is the so-called "bullwhip effect". It means that the demand information of from the downstream node along to the upstream node will be tremendous distorted.

Lee at el.[7,8,9] identified five main causes of the bullwhip: the use of demand forecasting, batch ordering, lead time, price fluctuation, and shortage gaming. Chen et al.[10,11] quantified the bullwhip effect for a simple supply chain model with the exponential smoothing forecasting. The ordering law in Chen's papers is the most popular replenishment rule, i.e. order up to level. They demonstrated that the bullwhip effect can be reduced by way of centralized customer demand information. Recently, Disney et al. [12] presented an analytic solution to the bullwhip effect for a specific ordering rule. These authors are all devoted to achieve the identical goals i.e. to eliminate the bullwhip effect and improve the customer satisfaction. However, if the customer demands are non-stationary time series, the classical process control results sometimes to worse performances.

Minimum variance control (MVC) has been a mature control scheme in the area of stochastic control. It has been widely used in industrial applications (e.g., C. Bordons et. al.[13]) On the other hand, it is very clear that the customer demand can be viewed as a stochastic process. The purpose of this work is to implement the principal of minimum variance control by modeling the customer demand as a stochastic process. In this work, we one step further, modify the objective function of the predictive by separating the inventory level into two parts, i.e., on the road and on the hand. By minimizing this novel objective function, the excess inventory and/or backorder can be furnished.

In general, the customer demand is stochastic with patterns. The major contribution of this work is to implement a general stochastic model to describe the demand of the customer. The stochastic model, in turn, can be implemented to predictive the future demand. The pattern of the demand can be further analyzed by using frequency domain analysis. The bullwhip effect of the unit can be analytically quantified. We further implement the general minimum variance control law, e.g., Soeterboek [14]. By tuning the parameters in the control law, the bullwhip effect can be effectively eliminated. The simulation results show that not only the inventory trajectory can be successfully tracked and hence excess inventory and backorder is minimized, but the ordering bullwhip effect can be effectively suppressed.

## 2. Theory

As we described in our previous paper (Lin et al.), a discrete time supply chain unit is proposed. Let's separate the general inventory control problem into the following parts.

### 2.1. The Balances of Material Flow and Information Flow

Without lose any generality, for simplicity, let's denote a simple decentralized supply chain system be three nodes, consisted of a upper stream node, a target node and a down stream node. Let *I(t)* be the inventory level of unit, *IP(t)* be the inventory position (the total amount of current inventory level plus the material delivered on the road from the upper streams) of the target node, $Y_U(t)$ be the product delivered from its upper level unit and $Y_D(t)$ be the material delivered to its down stream unit. Given current time *t*, a time delay of *L* (lead time) is assumed for all delivery actions so that goods dispatched at time *t* will arrive at time *t*+*L*. However, due to need for examination and administrative processing, this new delivery is only available to customer at *t*+*L*+1. We also assume that ordering information is communicated instantaneously. However, an order at time t will only be processed at time *t*+1. Let *O(t)* be the amounts of orders placed by the target node to its upstream. In order to describe the problem of excess inventory and backorders, let's separate *IP(t)* to two parts, namely $I_H(t)$ and $I_R(t)$. Then $I_H(t)$ and $I_R(t)$ are defined as the inventory level on hand and on road individually of the target unit. Hence, If we also assume that there is always enough stock in the upper stream, then:

$$Y_U(t) = z^{-1}O(t), \qquad Y_D(t) = U_C(t) \tag{1}$$

where $U_C(t)$ is the amount of orders from its downstream node, i.e., customer demand. The relation between inventory, order and demand is given by:

$$I_H(t) = \frac{1}{1-z^{-1}}\left(z^{-L-1}O(t) - U_C(t)\right) \tag{2}$$

$$I_R(t) = \frac{z^{-1}\left(1-z^{-L}\right)}{1-z^{-1}}O(t) \tag{3}$$

## 2.2. Minimum Variance Control

Consider the system described in the previous section, it is our objective to control the inventory level. The following objective function for a predictive controller can be set up by separating the inventory level into two parts, i.e. on the road and on the hand:

$$J = \sum_{i=m_H}^{i=n_H}\left(P(z^{-1})\hat{I}_H(t+i) - P_g SP_H(t+i)\right)^2 + \sum_{i=m_R}^{i=n_R}\left(P(z^{-1})\hat{I}_R(t+i) - P_g SP_R(t+i)\right)^2 + \rho\sum_{i=1}^{n_H-\hat{d}}\left(\Delta O(t)\right)^2 \quad (4)$$

The symbol "^" in Eq.(4) represents the predictive estimation of the variables. Here $d$ is the model time delay. $P(z^{-1})$ is a polynomial in $z^{-1}$, and $P_g$ is the gain of the polynomial $P(z^{-1})$. It performs a moving average of the predicted values of the controlled variables. $SP_H$ and $SP_R$ denote the set point of inventory on hand and road respectively. The objective function parameters $m_H$ (or $m_R$) and $n_H$(or $n_R$) are the minimum cost horizon and prediction horizon respectively. The parameter $\rho$ is the penalty factor to suppress the aggravated control actions. Eq.(4) consists of two controlled variables $I_H$ and $I_R$. Let customer demand $U_C(t)$ be a noise of the supply chain system. The following equations can be derived for the controlled variables $I_H$ and $I_R$. Assume that $I_H$ can be described by a general linear stochastic equation:

$$I_H(t+1) = \frac{z^{-d}B(z^{-1})}{A(z^{-1})}O(t) + \frac{C(z^{-1})}{D(z^{-1})}\xi(t) \quad (5)$$

Here $I_H$ is the controlled variable (CV) and $O$ is the manipulated variable (MV). The term $z^{-d}B(z^{-1})/A(z^{-1})$, with $B(z^{-1})$ and $A(z^{-1})$ being polynomials in $z^{-1}$ represents an input-output relation with a time delay $d$. The term $C(z^{-1})/D(z^{-1})\xi(t)$ with $C(z^{-1})$ and $D(z^{-1})$ being polynomials in $z^{-1}$, and $\xi(t)$ a white noise having mean zero, represents a disturbance to the system.

For the other controlled variable $I_R(t)$, by comparing to Eq.(3), we have no stochastic term in the model:

$$I_R(t+1) = \frac{Q(z^{-1})}{R(z^{-1})}O(t) \quad (6)$$

Since the supply chain is a simple logistic system, for simplicity, let's assume that:

(i) $n_H=m_H=L+1$, and $n_R=m_R=1$.

(ii) $P(z^{-1})=1+p_1z^{-1}$, $-1<p_1\leqq 0$

The optimization for $J$ can be solved to give the control law

$$O(t) = \frac{P_g SP_H(k+L+1) - \frac{F_{L+1}}{C}I_H(t)}{1+\rho\Delta+\frac{BDE_{L+1}}{AC}} + \frac{P_g SP_R(t+1) - \frac{V_1}{Q}I_R(t)}{1+\rho\Delta+\frac{BDE_{L+1}}{AC}} \quad (7)$$

For a decentralized supply chain system, the controlled variable is the inventory level $I(t)$ and the disturbance is the customer demand $U_C$. By comparing the system Eqs. (2) and (3) to the general input-output relations (5) and (6), we get

$$\frac{B(z^{-1})}{A(z^{-1})} = \frac{1}{\Delta} \quad (8)$$

$$\frac{Q(z^{-1})}{R(z^{-1})} = \frac{1-z^{-L}}{\Delta} \quad (9)$$

If we assume the customer demand takes the form of a time series as

$$U_C(t) = \frac{\Theta(z^{-1})}{\Phi(z^{-1})\Delta^r}\xi(t) \quad (10)$$

hence:

$$\frac{C(z^{-1})}{D(z^{-1})} = -\frac{\Theta(z^{-1})}{\Phi(z^{-1})\Delta^{r+1}} \quad (11)$$

and the control law in Eq.(7) becomes

$$O(t) = \frac{P_g SP_R(t+1) - z\left(P - \frac{\Delta}{1-z^{-L}}\right)I_R(t)}{1+\rho\Delta-\frac{E_{L+1}\Phi\Delta^r}{\Theta}} + \frac{P_g SP_H(t+L+1) - z^{L+1}\left(P+\frac{E_{L+1}\Phi\Delta^{r+1}}{\Theta}\right)I_H(t)}{1+\rho\Delta-\frac{E_{L+1}\Phi\Delta^r}{\Theta}} \quad (12)$$

Alternatively, one can also expressed the above equation in terms of customer demand

$$O(t) = \frac{\Delta P_g\left(SP_H(t+L+1)+SP_R(t+1)\right)}{P(z^{-1})\left(2-z^{-L}\right)+\rho\Delta^2}U_C(t) + \frac{z^{L+1}\left(P+\frac{E_{L+1}\Phi\Delta^{r+1}}{\Theta}\right)}{P(z^{-1})\left(2-z^{-L}\right)+\rho\Delta^2}U_C(t) \quad (13)$$

### 2.3. Performance Measure Criteria

In the problem of optimizing control of inventory level, two factors directly reflect the cost of the supply chain unit. One is the extra inventory level; the other is the backorders that measures the satisfaction of the customers. In this work, given a forecast horizon $\tau_H$ "optimal" controllers are obtained by minimizing an average excess inventory (*AEI*):

$$AEI = \frac{1}{\tau_H}\int_0^{\tau_H} I_H(t)dt, \quad \mathrm{I_H}(t) \ge 0 \quad (14)$$

or/and the average backorder order (*ABO*):

$$ABO = \frac{1}{\tau_H}\int_0^{\tau_H} \left|I_H(t)\right|dt, \ \mathrm{I_H}(t) < 0 \quad (15)$$

Subject to the following "bullwhip" constraint:

$$MR = \left|\frac{O}{U_C}\right| \le 1 \quad (16)$$

Based on the above criteria, a so-called minimum variance controller (MVC) derived in the next section can be tuned to different weightings of *AEI* and *ABO*.

### 2.4. Bullwhip Effect

By definition, the bullwhip effect can be measured by the ratio of order to its supplier to the demand from its customer node as

$$MR = \left|\frac{O}{U_C}\right| \quad (17)$$

In general, most supply chain researchers usually use a forecaster to predict customer demand. But in our paper, we use a minimum variance predictor instead of a forecaster to handle the customer demand. While having a lead time *L*, we can set $SP_H(t)=U_C(t)$ and $SP_R(t)=L \times U_C(t)$. Substitute the above two relations into Eq.(13) and calculate the magnitude ratio of $|O|/|U_C|$, thus we get

$$MR = \left|\frac{O}{U_C}\right| = \left|\frac{\Delta LP_g + z^{L+1}\left(P + \frac{E_{L+1}\Phi\Delta^{r+1}}{\Theta}\right)}{P\left(2 - z^{-L}\right) + \rho\Delta^2}\right| \quad (18)$$

Note that *MR* is not only a function of parameters setting of the controller, but a function of time series. However, Eq. (26) is basically a linear system; it is hence possible to implement the frequency domain analysis to find the *MR* at high frequencies if we assume $U_C$ is a stochastic process.

### 2.5. The Solution of the Optimizing Control Problem

Given the information and material balances of a single supply chain unit in section 2.1 and the stochastic demand model $U_C$, the direct solution of Eq.(10) can be shown in Eq.(13) with several tunable parameters. For simplicity, we only choose the most effective parameters ρ and $p_1$ in this study. Since the performance measure of *AEI* and *ABO* are directly related to the objective function, we direct solve the following optimization problem:

$$\min_{\rho, p_1} \ w \times AEI + (1-w) \times ABO \quad (19)$$
$$s.t. \ MR \le 1$$

and the order policy of Eq.(13). Note that in the above objective function, we put a weighting factor *w* into the objective function, since for different styles of supply chain unit may have their own consideration of the excess inventory and backorders.

## 3. Other Ordering Policies

This section reviews three different ordering policies to be compared with our approach in the next section. Note that there exist some other excellent different ordering policies, but we have compared some of them in our last work (Lin et al.)

### 3.1. Order up to Policy

The ordering rule in the textbook is as follows:

$$O(t) = SP(t) - IP(t) \quad (20)$$

Here *SP(t)* is the set point of inventory position, *IP(t)* is the inventory position. However in many literatures this equation is modified as the following equation to investigate the dynamic behaviors of the supply chain system.

$$O(t) = \Delta SP(t) + U_C(t) \quad (21)$$

### 3.2. Smoothing Order Policy (SOP)

As analyzed by J. Dejonckheere et al, the implementation of order up to policy, the bullwhip seems unavoidable. They hence proposed the following general replenishment rule:

$$O(t) = \tilde{U}_C(t) + \frac{1}{T_n}\left(\tilde{U}_C(t) - I_H(t)\right) + \frac{1}{T_w}\left(L\tilde{U}_C(t) - I_R(t)\right) \quad (22)$$

Where $T_n$ and $T_w$ are the parameters of the ordering decision rule, $\tilde{U}_C$ is the demand forecast,

$$\tilde{U}_C(t) = F_R(z^{-1})U_C(t) \quad (23)$$

where $F_R(z^{-1})$ is a forecaster. It is easily found that the net stock ($I_H$) plus products on order ($I_R$) equals inventory position ($IP$). In case $T_n$=$T_w$, then Eq.(31) will be reduced as

$$O(t) = \tilde{U}_C(t) + \frac{1}{T_n}\left((L+1)\tilde{U}_C(t) - IP(t)\right) \quad (24)$$

The readers will easily observe that the smoothing ordering rule is a controller model consisted of a feed-forward controller and a proportional controller. For simplicity, if $T_n$=$T_w$=1 the smoothing law will be reduced to the order up to policy.

### 3.3 PI Control Policy

The authors (Lin et al.) proposed a PI controller for supply chain unit since PI controller is a traditional controller that guarantees no off-set.

$$O(t) = K_C\left(1 + \frac{1}{\tau\Delta}\right)\left(SP(t) - IP(t)\right) \quad (25)$$

where $K_C$ is the proportional gain, and $\tau$ is the integral constant. Of course, when $K_C$ =1 and $1/\tau$=0, the PI controller will be also changed to the order up to policy.

## 4. Examples

A case is considered in this work to demonstrate the capabilities of this approach to track the change of demand and to avoid the bullwhip effect. We compare our approach with other popular approaches such as order up to level, and smooth ordering rule proposed by Dejonckeere et al. which have been shown very effective to eliminate the bullwhip effect.

In many real supply chain cases,customer demands exhibit non-stationary behavior due to many particular events such as season changes, new designs…etc. Such system can be also modeled by Eq.(10) by implementing $r \neq 0$.

For example: $r=1$, $\Phi(z^{-1}) = 1 - 0.6z^{-1}$, and $\Theta(z^{-1}) = 1$.

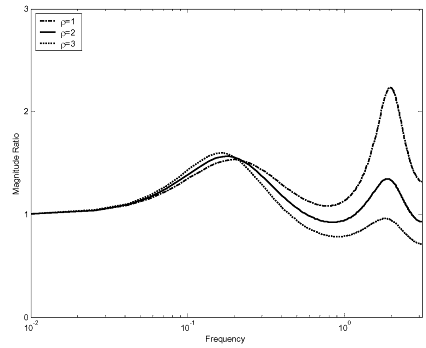


Figure 2. The magnitude ratio of order to demand using various weighting factors $\rho$ with the same value of $p_1$=-0.8

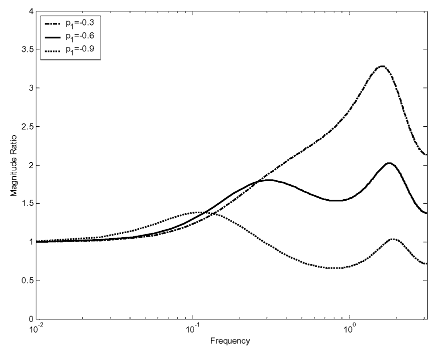


Figure 3. The magnitude ratio of order to demand using various values of $p_1$ with the same penalty factor $\rho$ =2.

Similarly, the magnitude ratio of $|O|/|U_C|$ with the different weighting factors ($\rho$ =1,2,3) for the non-stationary demand process as depicted in Figure 2. And Figure 3 also gives frequency domain analysis for some different values of $p_1$. From the above two Figures 2 and 3, we obtain the following information: The bigger $\rho$ or smaller $p_1$ will reduce bullwhip effect. or a MVC controller, Figure 4 give the contour plots of both $\Psi$ and MR as a function of $\rho$

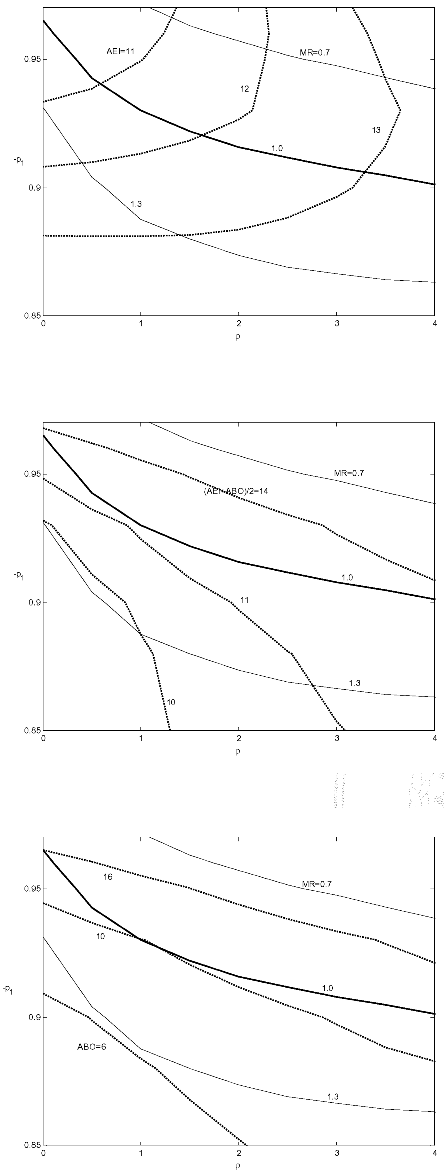


Figure 4 Contour diagram of MR(dot lines) and ABO (solid lines) as functions of $\rho$ and $p_1$of a MVC, while (a)w=1.0 (b)w=0.5 (c)w=0.0

and $p_1$. Figure 5 displays the dynamic simulation results of a supply chain by using a predictive controller with $\rho$=0 and $p_1$=-0.96 for the scenery w=0. Table 1 gives the optimal simulation results of the objective function by implementing four different controllers. The MVC controller approach appears superior to other approaches as shown in Table 1.

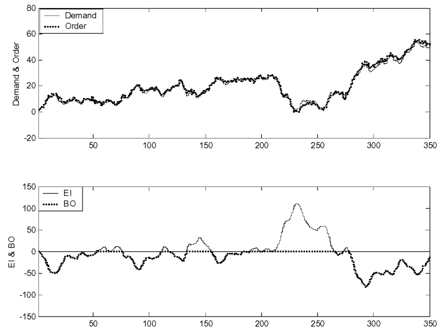


Figure 5. Dynamic simulation results of a supply chain unit for a MVC using $\rho$ =0 and $p_1$=-0.96, while w=1.0 for a non-stationary demand process.

## 5. Conclusion

A very general approach for supply chain ordering policy for a known demand pattern is derived. The novel approach includes:

(1) A general discrete model of the demand.
(2) A minimum variance ordering policy.
(3) A closed-loop frequency domain analysis of the supply chain unit.
(4) A fine tune rule that can avoid the bullwhip effect and minimize the excess inventory and/or customer dissatisfaction.

The whole approach is shown to be valid and superior to existed approaches through real time simulations on both stationary and non-stationary cases.

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Table1: Non-stationary demand

| Weighting factor | Control law | Parameters | MR | AEI | ABO |
|---|---|---|---|---|---|
| | Order up to policy | | 2.04 | 14.44 | 4.00 |
| W=1.0 | Smoothing ordering rule | Tn=7.80 Tw=4.00 | 1.00 | 14.55 | 21.89 |
| | Proportional Integral control | Kc=0.150 $\tau$ =37.5 | 1.01 | 19.69 | 17.61 |
| | Predictive control | $\rho$ =0.00 $p_1$=-0.96 | 1.00 | 10.77 | 15.76 |
| w=0.5 | Smoothing ordering rule | Tn=7.80 Tw=4.00 | 1.00 | 14.55 | 21.89 |
| | Proportional Integral control | Kc=0.146 $\tau$ = 36.5 | 1.00 | 19.92 | 18.09 |
| | Predictive control | $\rho$ =0.60 $p_1$=-0.94 | 1.00 | 11.95 | 11.26 |
| w=0.0 | Smoothing ordering rule | Tn=7.80 Tw=4.00 | 1.00 | 14.55 | 21.89 |
| | Proportional Integral control | Kc=0.125 $\tau$ =25.0 | 1.00 | 23.64 | 16.92 |
| | Predictive control | $\rho$ =1.00 $p_1$=-0.93 | 1.00 | 12.46 | 9.91 |

## 6. Reference:


1. D.R. Towill, Dynamic analysis of an inventory and order based production control system. *International Journal of Production Research* **20**(**1982**) 671-687.
2. E. Perea, I. Grossmann, E. Ydstie, and T. Tahmassebi, Dynamic modeling and classical control theory for supply chain management. *Computers & Chemical Engineering* **24** (**2000**) 1143-1149.
3. E. Perea-López, I.E. Grossmann, B.E. Ydstie,. and T. Tahmassebi, Dynamic modeling and decentralized control of supply chains. *Industrial & Engineering Chemistry Research*, **40** (**2001**) 3369-3383.
4. J. Dejonckheere, S.M. Disney, M.R. Lambrecht, and D.R. Towill, Transfer function analysis of forecasting induced bullwhip in supply chains. *International Journal of Production Economics* **78** (**2002**) 133-144.
5. J. Dejonckheere, S.M. Disney, M.R. Lambrecht, and D.R. Towill, Production, Manufacturing and Logistics - Measuring and avoiding the bullwhip effect: A control theoretic approach. *European Journal of Operational Research* **147** (**2003**) 567-590.
6. P.H. Lin, D.S.H. Wong, S.S. Jang, S.S. Shieh, and J.Z. Chu , Controller design and reduction of bullwhip for a model supply chain using z-transform analysis. *Journal of Process Control* **14** (**2004**) 487-499
7. H.L. Lee, K.C. So, and C.S. Tang, The value of information sharing in a two-level supply chain. *Management Science* **46**(**2000**) 626-643.
8. H.L. Lee, V. Padmanabhan, and S. Whang, The bullwhip effect in supply chains. *Sloan Management Review* **38** (**1997**) 93-102.

9. H.L. Lee, V. Padmanabhan, and S. Whang, Information distortion in supply chain: The Bullwhip effect. *Management Science* **43**(**1997**) 546-558.
10. F. Chen, Z. Drezner, J.K. Ryan, and D. Simchi-Levi, Quantifying the bullwhip effect in a simple supply chain: The impact of forecasting, lead times, and information. *Management Science* **46** (**2000)** 436-443.
11. F. Chen, J.K. Ryan, and D. Simchi-Levi, The impact of exponential smoothing forecasts on the bullwhip effect. *Naval Research Logistics* **47** (**2000**) 269-286.
12. S.M Disney, and D.R. Towill, On the bullwhip and inventory variance produced by an ordering policy. *Omega* **31**(**2003**) 157-167.
13. C. Bordons, and J.R. Cueli, Predictive controller with estimation of measurable disturbance. Application to an olive oil mill. *Journal of Process Control* **14**(**2004**) 305-315.
14. Ronald Soeterboek, *Predictive Control: A Unified Approach*. Prentice Hall International (**1992**).